The Size of the Smallest Strong Critical Set in a Latin Square

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Abstract A critical set in a latin square is a set of entries in a latin square which can be embedded in only one latin square. Also, if any element of the critical set is deleted, the remaining set can be embedded in more than one latin square. A critical set is strong if the embedding latin square is particularly easy to find because the remaining squares of the latin square are "forced" one at a time. A semi-strong critical set is a generalization of a strong critical set. It is proved that the size of the smallest strong or semi-strong critical set of a latin square of order n is $\lfloor n^2/4 \rfloor$. An example of a critical set that is not strong or semi-strong is also displayed. It is also proved that the smallest critical set of a latin square of order 6 is 9.

1. Introduction

This paper deals with critical sets in latin squares. A latin square, L, of order n is a n x n array with elements chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. For example, let us index the rows and columns of the array by the set {0,1,2,...,n-1} . If the integer i+j (modulo n) is placed in position (i,j) of the array, then the result is a latin square, called a back circulant latin square of order n. For convenience, we will sometimes refer to the latin square, L, as a set of ordered triples, (i,j;k) where this notation means that symbol k is in row i and column j. The ordered triple is referred to as the entry of the latin square.

A partial latin square, P, of order n is an n x n array with the elements chosen from a set N of size n, such that each element of N occurs at most once in each row and column. So P may contain a number of empty cells. Note that P does not have to complete to or embed into a latin square. The partial latin square may be referred to as a set of triples where only the triples corresponding to non-empty squares are listed. A partial latin square, P, of order n where $P = \{ (i,j;k) \mid i,j,k \in \{ 0,1,...,n-1\} \}$, is said to be uniquely completable if there is only one latin square of order n which has symbol k in position (i,j) for each

$(i,j;k) \in P$. A critical set in a latin square, L, is a partial latin square which has unique completion to L and all proper subsets of P complete to at least two distinct latin squares. As an example, consider the critical set { (0,0;0), (0,1;1), (1,0;1), (3,4;2), (4,3;2), (4,4;3)} in the latin square of order 5 where the (i,j)th term is i+j modulo 5. Since it is much easier to comprehend these sets as two dimensional objects, they are presented in Figure 1. The - indicates an empty square.

```
0 1 2 3 4      0 1 - - -
1 2 3 4 0      1 - - - -
2 3 4 0 1      - - - - -
3 4 0 1 2      - - - - 2
4 0 1 2 3      - - - 2 3
```

Figure 1 A latin square (left) and a critical set in that latin square (right)

In the late 70's, Curran and van Rees [4] and Smetaniuk [10] introduced critical sets as objects of curiosity. In the first half of the 80's, Stinson and van Rees [11] did some basic research on sizes of critical sets and Colbourn, Colbourn and Stinson [1] proved that the computational complexity of recognizing whether a partial Latin square has a unique completion is NP-complete. In the 90's, Seberry [9] showed that critical sets have applications in agricultural research and in cryptography. This stimulated a series of papers: Cooper, Donovan and Seberry[2], Donovan, Cooper, Nott and Seberry[6], Cooper, McDonough and Mavron[3] and Keedwell[7].

Define scs(n) to be the cardinality of the smallest critical set in any latin square of order n. Much research has been done on this function but little is known about it. Curran and van Rees [4] proved the obvious bound that $scs(n) \geq n-1$. This was easily improved by Cooper, Donovan and Seberry to $scs(n) \geq n$ for $n \geq 4$. With more work, Cooper, McDonough and Mavron proved that $scs(n) \geq n+1$ for $n \geq 5$. The upper bound improves even more slowly. Curran and van Rees showed that $scs(n) \leq \lfloor n^2/4 \rfloor$ by showing that a set isotopic to the following set P was or contained a critical set in the back circulant latin square of order n: $P = \{ (i,j;i+j) \mid i=0,\ldots, \lfloor (n-2)/2 \rfloor \text{ and } j=0,\ldots,\lfloor (n-2)/2 \rfloor -i\} \cup \{ (i,j;i+j) \mid i=\lfloor (n+2)/2 \rfloor ,\ldots,n-1 \text{ and } j=(\lfloor (3n)/2 \rfloor -i),\ldots,n-1\}$. When n is even, Curran and Van Rees showed that P actually was a critical set. Cooper, Donovan and Seberry [2] showed that for odd n, the set P was actually a critical set.

This leaves a big gap between the upper and lower bounds. We record this in the following theorem.

Theorem 1.1. $n+1 \leq scs(n) \leq \lfloor n^2/4 \rfloor$ for $n \geq 5$.

2. Strong and Semi-Strong Critical Sets

In the paper by Donovan, Cooper, Nott and Seberry [6], they defined an object easier to deal with than a critical set. Let L be a latin square of order n, based on the set N. Let L contain a critical set A. They defined the set A to be a strong critical set if there exists a set $\{P_1,...P_m\}$ of $m=n^2-|A|$ partial latin squares, of order n, which satisfy the following properties:

1) $L \supset P_m \supset P_{m-1} \supset P_2 \supset P_1 = A$;
2) for any i, $1 \leq i \leq m-1$, $P_i \cup \{ (r_i,c_i ; e_i) \} = P_{i+1}$ and $P_i \cup \{ (r_i,c_i ; e) \}$ is not a partial latin square for any $e \in N \setminus \{ e_i\}$.

For example, the critical set of the back circulant latin square previously displayed is a strong critical set. $(r_1,c_1 ; e_1)$ must be (0,4 ; 4) because row 1 of the critical set contains the elements 0 and 1 and column 4 contains the elements 2 and 3 so the entry in position (0,4) is forced to be (0,4 ; 4). It is trivial to check that at any stage of the procedure there is at least one entry that is forced in P_i in exactly this way. This can be expressed accurately in the following lemma. Let E_i be the set of elements in row r_i and column c_i in P_i.

Lemma 2.2 $| E_i | = n-1$.

Proof. The only way $P_i \cup \{(r_i,c_i ; e)\}$, where $e \in N \setminus \{ e_i\}$, cannot be a partial latin square is if row r_i or column c_i of P_i already contain the element e. ❑

It should be noted that there may be and usually are many sequences of P_i for a specific strong critical set. The critical sets of Curran and van Rees, mentioned previously, are strong critical sets. Indeed, all critical sets displayed in the literature are strong critical sets. Cooper, Donovan and Seberry [2] go on to prove the following lemma.

Lemma 2.3 Let A be a strong critical set in a latin square L of order n, then $| A | \geq (4n-6)/3$.

This lemma can be improved by the following theorem.

Theorem 2.4 Let A be a strong critical set in a latin square L of order n, then $| A | \geq \lfloor n^2/4 \rfloor$.

Proof. From the sequence P_1, P_2, ... P_m pick a subsequence $P_{(j,1)}$, $P_{(j,2)}$, … $P_{(j,\lfloor n/2 \rfloor)}$, where (j,1) is 1 and (j,2) is the first subscript i such that $r_i \neq r_{(j,1)}$ and $c_i \neq c_{(j,1)}$. In general, (j,t) is the first subscript i such that $r_i \neq r_{(j,1)}$, $r_{(j,2)}$, ... or $r_{(j,t-1)}$ and $c_i \neq c_{(j,1)}$, $c_{(j,2)}$, ... or $c_{(j,t-1)}$. If there is no such (j,t) because all the remaining squares are filled in then $|A| > \lfloor n^2/4 \rfloor$. So let us assume that the (j,t)'s all exist. Since $|E_{(j,1)}|$ = n-1, A must contain at least n-1 entries in the union of entries from row $r_{(j,1)}$ and column $c_{(j,1)}$. Now $|E_{(j,2)}|$ = n-1. Of these n-1 elements, at least n-1-2 of them are not in row $r_{(j,1)}$ or in column $c_{(j,1)}$. So A contains at least (n-1)+(n-3) entries from row $r_{(j,1)}$, row $r_{(j,2)}$, column $c_{(j,1)}$ and column $c_{(j,2)}$. Also, $|E_{(j,t)}|$ = n-1. Of these n-1 elements, n-1-2(t-1) of them are not in rows $r_{(j,1)}$, $r_{(j,2)}$,...,$r_{(j,t-1)}$ or in columns $c_{(j,1)}$, $c_{(j,2)}$,...,$c_{(j,t-1)}$. Therefore A contains at least (n-1) + (n-3) + ... + (n-1- 2(t-1)) entries in rows $r_{(j,1)}$, $r_{(j,2)}$, ... $r_{(j,t)}$ and columns $c_{(j,1)}$, $c_{(j,2)}$, ... $c_{(j,t)}$. If we consider all the $E_{(j,t)}$'s, we obtain $|A| \geq \lfloor n^2/4 \rfloor$. ❑

This allows us to end the debate on the size of the smallest strong critical set.

Corollary 2.5 The cardinality of the smallest strong critical set of a latin square of order n is $\lfloor n^2/4 \rfloor$.

Proof. Note that $\lfloor n^2/4 \rfloor$ is the size of the Curran and van Rees strong critical sets and compare with previous theorem. ❑

The following is a critical set that is not a strong critical set and the latin square which is its unique completion.

```
- 2 3 4 - -      1 2 3 4 6 5
3 - - - - 4      3 1 2 6 5 4
- - - 5 - -      2 3 1 5 4 6
- 4 6 - - 1      5 4 6 2 3 1
- 6 - - 2 -      4 6 5 1 2 3
6 - - - 1 -      6 5 4 3 1 2
```

It is easy to check that the set on the left has unique completion and each element is needed for unique completion. However, if each empty square of the partial latin square on the left is checked, it will be noted that there are at least two choices for every element. This means that this critical set is not strong.

However, if we take the conjugate of this latin square and the conjugate of the critical set where the conjugate takes the triple (a,b;c) and replaces it with (b,c;a), we see that the conjugate of the critical square is a strong critical set for the conjugate of the latin square.

This can happen because, as is pointed out in Colbourn, Colbourn & Stinson[1], there are three ways that an entry can be forced. One can look at the row and column to find that a particular element has to go into the intersection of the row and column as is used in the definition of strong critical set. However, one can look at a row and a particular element to see which column the element must go into in that row i.e. there are a total of n-1 distinct columns in that row that are either filled or that column already contains that particular element. Similarly, one can use a column and an element to find that there is only one row in that column that could contain that element. This leads us to the following definition. Let L be a latin square of order n, based on the set N. Let L contain a critical set A. Then define the set A to be a semi-strong critical set if there exists a set $\{P_1,...P_m\}$ of $m=n^2-|A|$ partial latin squares, of order n, which satisfies the following properties:

1) $L \supset P_m \supset P_{m-1} \supset P_2 \supset P_1 = A$;

2) for any i, $1 \le i \le m-1$, $P_i \cup \{(r_i,c_i; e_i)\} = P_{i+1}$ such that one of $P_i \cup \{(r_i,c_i; e)\}$ or $P_i \cup \{(r_i,c ;e_i)\}$ or $P_i \cup \{(r,c_i; e_i)\}$ is not a partial latin square for any $e \in N \setminus \{ e_i\}$ or $c \in N \setminus \{ c_i\}$ or $r \in N \setminus \{ r_i\}$, respectively. We will let C_i be the set of columns in row r_i of P_i that are either non-empty or if empty that column contains e_i. Let R_i be the set of rows in column c_i of P_i that are either non-empty or if empty that row contains e_i. It is easy to see that Lemma 2.2 can be generalized to the following.

Lemma 2.6 Either $|E_i| = n-1$ or $|R_i| = n-1$ or $|C_i| = n-1$.

We now generalize Theorem 2.4 to semi-strong critical sets using the same techniques as before.

Theorem 2.7 Let A be a semi-strong critical set in a latin square L of order n, then $|A| \ge \lfloor n^2/4 \rfloor$.

Proof. Consider the sequence P_1, P_2, ... P_m. Let $(r_i,c_i ; e_i)$ be forced by consideration of X_i where X_i is one of R_i, C_i or E_i. If X_i can be forced in more than one way, let X_i be one of R_i, C_i or E_i at random but specified for this argument. From the sequence P_1, P_2, ... P_m pick a subsequence $P_{(j,1)}$, $P_{(j,2)}$,

... $P_{(j,\lfloor n/2 \rfloor)}$, where (j,1) is 1 and (j,2) is the first subscript i such that $r_i \neq r_{(j,1)}$, $c_i \neq c_{(j,1)}$ and $e_i \neq e_{(j,1)}$. In general, (j,t) is the first subscript i such that $r_i \neq r_{(j,1)}$, $r_{(j,2)}$, ... or $r_{(j,t-1)}$, $c_i \neq c_{(j,1)}$, $c_{(j,2)}$, ... or $c_{(j,t-1)}$ or $e_i \neq e_{(j,1)}$, $e_{(j,2)}$, ... or $e_{(j,t-1)}$. Suppose that there is no such (j,t). Then there are (n-1) + (n-3)+ ... +(n–1–2(t–1)) elements counted so far in A. There must be (n-t)(n-t)-t(n-t) entries in the remaining rows and columns of $P_{(j,t)}$ which are not in row r_i, or column c_i or have element e_i where i ε { (j,1), (j,2), ... (j,t-1) }. Therefore $|A| \geq t(n+1-2t)+t(t-1)+(n-2t)(n-t)$ or $|A| \geq (n-t)^2$. Since $t \leq \lfloor n/2 \rfloor$, this would imply the theorem so let us assume that (j,t) always exists.

At the p'th step, $|X_{(j,p)}| = n-1$. Of the $X_{(j,1)}$, $X_{(j,2)}$, ... $X_{(j,p-1)}$ let u of them be R_i, let v of them be C_i and let w of them be E_i. Clearly u+v+w=2(p-1). That means that the entry $(r_{(j,p)},c_{(j,p)} ; e_{(j,p)})$ does not occur in v+w specific rows, u+w specific columns or use u+v specific elements. Let $X_{(j,p)}$ be a $R_{(j,p)}$ (the proof for $C_{(j,p)}$ and $E_{(j,p)}$ are similar). So $|R_{(j,p)}| = n-1$. The entry added to the partial latin square $P_{(j,p-1)}$ is $(r_{(j,p)}, c_{(j,p)} ; e_{(j,p)})$. This entry is forced because $(r_{(j,p)}, c_{(j,p)})$ is the only square $(r, c_{(j,p)})$ in column $c_{(j,p)}$ of $P_{(j,p-1)}$ that is empty and does not have element $e_{(j,p)}$ in row r. Now, of the (n-1) squares in column $c_{(j,p)}$ in $P_{(j,p-1)}$, at most u+v are filled with elements $e_{(j,s)}$, $s \leq p-1$; at most v+w are filled in at square $(r_{(j,s)},c_{(j,p)})$, $s \leq p-1$; at most u+w are empty squares $(r,c_{(j,p)})$, where row r contains an element $e_{(j,s)}$, $s \leq p-1$. So there must be at least n-1-2(u+v+w) = n-1-2(p-1) entries in A, the critical set, either as entries $(r,c_{(j,p)} ; e_{(j,s)})$ or entries $(r,c ; e_{(j,p)})$ where square $(r,c_{(j,p)})$ is empty.

Considering this for all X_j, gives $|A| \geq (n-1)+(n-3)+...(+1 \text{ or } 0)$. Therefore $|A| \geq \lfloor n^2/4 \rfloor$. ❑

Since the critical sets of Curran and van Rees are semi-strong we can give the following corollary.

Corollary 2.8 The cardinality of the smallest semi-strong critical set of a latin square of order n is $\lfloor n^2/4 \rfloor$.

Colbourn, Colbourn and Stinson [1] claimed to display a critical set that could not be completed by forcing. Unfortunately, there are four typographical errors in the paper. The first three are mis-typed entries in the critical set and the fourth is that the set displayed is not a critical set but it is a set (when suitably corrected) with unique completion that can not be forced. Figure 2 is the set with unique completion from Colbourn et al. and the latin square it completes to: (t=ten, e=eleven and v=zero). The critical set is in plain type (underlined or not) and the latin square completion is in bold type.

e	8	2̲	**3**	4	7	9̲	t	v	5	**1**	6̲
t̲	**4**	9	**1**	8̲	**6**	e	v	3	2	7̲	5
9	e̲	**5**	t	**2**	v̲	**7**	3̲	8̲	6	4	1̲
3	t	**1**	8̲	e̲	**5**	v	4	9̲	7̲	6̲	2
7	**2**	6̲	e̲	1̲	9	**4**	5̲	**t**	v̲	8̲	3
v̲	**6**	e	**4**	9̲	**3**	**2**	**8**	5	1̲	t̲	7̲
8̲	9	**7**	6̲	**5**	**1**	3̲	e	**2**	4̲	**v**	t̲
2̲	3̲	v	7	t̲	**8**	1̲	**6**	e̲	**9**	5̲	4̲
5̲	v̲	4̲	2̲	6̲	e̲	**8**	1̲	**7**	**t**	3	9̲
4̲	7	t̲	5	3	2̲	6̲	**9**	1̲	**8**	**e**	**v**
1	5̲	3̲	v̲	7	4̲	**t**	2̲	6̲	**e**	**9**	**8**
6	1̲	8	9̲	**v**	t	5̲	7̲	4̲	3̲	**2**	**e**

Figure 2

In order to get a critical set, we deleted elements from their unique completion set until the set was a critical set. The critical set is the set of underlined elements. It has cardinality 63. Note that the underlined set forces several squares of the latin square to be filled in just one way. But the elements shown in bold type are not forced by the elements shown in plain type, and therefore are also not forced by the underlined elements. Hence the underlined set is a critical set that is not strong or semi-strong. This is the first such example in the literature. It was checked for minimality by computer.

3. Determination of scs(6)

The previously known values of scs(n), the cardinality of the smallest critical set in a latin square of order n, are shown in Table 3.1.

n	1	2	3	4	5	6	7
scs(n)	0	1	2	4	6	7-9*	8-12

[* Here we will show that scs(6)=9]

Table 3.1

By a combination of analytical results and exhaustive computer searches, we have been able to verify that scs(6)=9. There are twelve main classes of latin squares of order 6 listed in Denes and Keedwell [5]. Donovan, Cooper, Nott, and Seberry [6] have shown that it is sufficient to determine the size of the smallest critical set in one latin square from each of these classes.

We require the following two lemmas.

Lemma 3.1 If a latin square L contains a latin subsquare S of order $n \geq 2$, then any critical set C of L must contain at least scs(n) elements from S.

Proof. If there are fewer than scs(n) elements from S in C, then there will be more than one way to complete the subsquare S, leading to more than one possible completion of L itself, which contradicts the assertion that C is a critical set.

Corollary 3.2 If a latin square L contains a latin subsquare S of order 2, then any critical set C of L must contain at least one element from S.

Corollary 3.3 If a latin square L contains a latin subsquare S of order 3, then any critical set C of L must contain at least two elements from S (and these elements must force a unique completion of S).

Classes 6.1.1, 6.2.1, 6.4.1, 6.5.1, and 6.10.1 (as listed in [5]) of latin squares of order 6 each contain 9 disjoint latin subsquares of order 2, which immediately gives $scs(6) \geq 9$ for these classes by Corollary 3.2.

1	2	3	4	5	6
2	3	1	6	4	5
3	1	2	5	6	4
4	6	5	1	2	3
5	4	6	2	3	1
6	5	4	3	1	2

Latin square 6.9.1

It can be proven that latin square 6.9.1 (above) has no critical sets with fewer than 9 elements as follows. Assume that latin square 6.9.1 has 8 elements in its critical set. Since the square is made up of 4 disjoint 3x3 latin subsquares, each of these subsquares must contain exactly scs(3)=2 elements in the critical set and those two elements must be a critical set for the subsquare (by Corollary 3.3). Further, this square contains 9 2x2 latin subsquares all involving the element 4. But we can have at most one 4 from each 3x3 subsquare in any crtitical set since a critical set of size 2 in a 3x3 subsquare must consist of 2 distinct elements in different rows and columns. No matter which of the 9 pairs of 4's we pick, say (1,4) and (4,1), we are then forced to pick elements in the other 3x3 subsquares that will give a contradiction. For example, by looking at 2x2 subsquares, we must pick

one of (2,2) and (5,5)	one of (3,2) and (5,6)
one of (2,3) and (6,5)	one of (3,3) and (6,6)

No choice gives critical sets for the two subsquares. Hence latin square 6.9.1 must have a critical set of size at least 9.

In order to complete the determination that scs(6)=9, a computer program was used to rule out the existence of smaller critical sets in the latin squares from the remaining classes (6.3.1, 6.6.1, 6.7.1, 6.8.1, 6.11.1, and 6.12.1) by using exhaustive backtrack search techniques. A program was written which will take a particular latin square L, and search for all possible critical sets S of a given size K in that latin square. (It verifies that the set S may be embedded only in the given square, but does not check to that no subsets of S also have this property. This is easily verified, and in most cases it is known in advance.)

The program operates by generating all possible candidate sets C of size K which satisfy the following conditions:

1. If S is a 2x2 latin subsquare of L, then at least one element of S must appear in C.
2. C must contain elements from at least N-1 of the rows of L, at least N-1 of the columns of L, and at least N-1 distinct elements, where N is the size of L.

Each candidate is then tested to see if it may be embedded in any latin square other than L. If not, then it is a critical set and is written.

In order to take advantage of the automorphisms of L, and also to provide checkpoints and progress indicators, the program provides a mechanism for manually directing the higher levels of the search. The user may specify a file containing a list of individual cases, and each case may force certain elements to be included in C, and also prevent other elements from being included in C. A specification of the form “U r c” will guarantee that L[r,c] is included (Used) in C, and a specification of the form “R r c” will prevent (Reject) L[r,c] from being included in C. Each line in the file is treated as a separate case and may contain any number of such specifications.

For example, in the search for critical sets in the following latin square L (6.11.1 in Denes and Keedwell):

```
1 2 3 4 5 6
2 1 4 5 6 3
3 4 2 6 1 5
4 5 6 2 3 1
5 6 1 3 2 4
6 3 5 1 4 2
```

it was determined that L[1,1] and L[2,2] are in the same orbit of the automorphism group, as are L[1,2] and L[2,1], by using the Groups and Graphs program written by W.L.Kocay [8]. Since one of these four elements must be chosen (they form a 2x2 subsquare of L), only 2 cases need be considered: 1) L[1,1] is used in C; 2) Neither L[1,1] nor L[2,2] are used in C, but L[1,2] is used in C. Note that case 1) does not preclude L[2,2] from also being included in C, nor does case 2) preclude L[2,1] from also being included in C. Since without direction the program would try to use all four elements of this subsquare, the specification of these two cases immediately reduces the running time of the program by half. Each of these two cases may now be treated similarly. For example, using case 1), once L[1,1] has been chosen, there is still an order 8 automorphism in L, and L[3,3] and L[4,4] are in the same orbit, as are L[3,4] and L[4,3]. These elements form another 2x2 latin subsquare, and so the same technique may be applied again. After a third application of this technique in every subcase, the search has been pruned from 64 cases to only 13. Here is the complete file of cases that was used in the search for critical sets in the above latin square:

```
U 1 1 U 3 3 U 5 4
U 1 1 U 3 3 R 5 4 R 4 5 U 4 4
U 1 1 U 3 3 R 5 4 R 4 5 R 4 4 U 5 5
U 1 1 R 3 3 R 4 4 U 3 4 U 4 4
U 1 1 R 3 3 R 4 4 U 3 4 R 4 4 U 5 4
U 1 1 R 3 3 R 4 4 U 3 4 R 4 4 R 5 4 U 5 5
U 1 1 R 3 3 R 4 4 U 3 4 R 4 4 R 5 4 R 5 5 U 4 5
R 1 1 R 2 2 U 1 2 U 3 3 U 4 4
R 1 1 R 2 2 U 1 2 U 3 3 R 4 4 R 6 6 U 4 6
R 1 1 R 2 2 U 1 2 U 3 3 R 4 4 R 6 6 R 4 6 U 6 4
R 1 1 R 2 2 U 1 2 R 3 3 R 4 4 U 3 4 U 5 5
R 1 1 R 2 2 U 1 2 R 3 3 R 4 4 U 3 4 R 5 5 R 6 6 U 5 6
R 1 1 R 2 2 U 1 2 R 3 3 R 4 4 U 3 4 R 5 5 R 6 6 R 5 6 U 6 5
```

Similar techniques were used to reduce the size of the search for critical sets in the latin squares 6.3.1, 6.6.1, 6.7.1, 6.11.1 (above), and 6.12.1. In each of these cases, no critical sets of size 7 or 8 were found. The program which was used (written in Pascal), together with all of the data files, files of case specifiers, and files of results, are available for inspection at the ftp site ftp.cs.umanitoba.ca in the directory pub/bate/critset (The URL is ftp://ftp.cs.umanitoba.ca/pub/bate/critset/).

We now have

Theorem 3.4 scs(6) = 9.

Since there is now more known about scs(n), it is an appropriate time for the following conjecture.

Conjecture 3.5 $scs(n) = \lfloor n^2/4 \rfloor$.

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