SIAM J. COMPUT.
Vol. 4, No. 4, December 1975

THE ENUMERATION OF GENERALIZED DOUBLE STOCHASTIC NONNEGATIVE INTEGER SQUARE MATRICES*

D. M. JACKSON† AND G. H. J. VAN REES‡

Abstract. The problem of enumerating generalized double stochastic integer square matrices is considered. The superposition theorem is used in conjunction with Schur functions to obtain the counting series for the 5×5 and 6×6 cases.

Key words. enumeration, stochastic matrices, generating functions, partitions, Schur functions, symmetric group

1. Introduction. Let $H_m(n)$ be the number of $m \times m$ matrices over $\mathbb{N} \equiv \{0, 1, 2, \cdots\}$ with line sum $n \in \mathbb{N}$. The purpose of this paper is to give the ordinary generating functions $\Phi_5(t)$, $\Phi_6(t)$ for the sequences $\{H_5(n)\}$ and $\{H_6(n)\}$ respectively with the anticipation that these additional sequences may be of service in determining the relationship among the $\Phi_i(t)$. The sequences for $\{H_3(n)\}$, $\{H_4(n)\}$ have been given by Sloane [1], together with references to the papers in which they first appear. The theorems supporting the computational method are given in § 2, while § 3 contains examples and a special case. The initial segments of sequences $\{H_i(n)\}$ and the generating functions $\Phi_i(t)$ are tabulated in § 4 for $i = 2, 3, 4$ (for completeness) and for $i = 5, 6$. Use has been made of Theorem 2, which is amenable to automatic computation.

2. Main theorem. The following result permits the computation of $H_m(n)$ by polynomial interpolation. The symmetry relation reduces the number of values of n at which $H_m(n)$ need be computed for interpolatory purposes.

CONJECTURE (Anand–Dumir–Gupta [2]).

(i) *$H_m(n)$ is a polynomial in n of degree $(m-1)^2$.*

(ii) $H_m(n) = (-1)^{m-1} H_m(-m-n)$.

Proof. See Stanley [3].

Values of $H_m(n)$ may be computed according to the following theorem.

THEOREM 1. $H_m(n) = N(h_n^m * h_n^m)$, *where*

(i) *h_n is the cycle index polynomial in the indeterminates $x_1 \cdots x_n$ for the symmetric group S_n;*

(ii) *if $A(\mathbf{x})$, $B(\mathbf{x})$ are two multivariate polynomials in $x_1 \cdots x_n$ such that*

$$A(\mathbf{x}) = \sum_{(\mathbf{i})} a_\mathbf{i} \mathbf{x}^\mathbf{i},$$

$$B(\mathbf{x}) = \sum_{(\mathbf{i})} b_\mathbf{i} \mathbf{x}^\mathbf{i},$$

where the summation is over all partitions $(\mathbf{i}) = 1^{i_1} 2^{i_2} \cdots n^{i_n}$ of n, then the

* Received by the editors June 17, 1974. This work was supported by the National Research Council of Canada.

† Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. This work was done while the author was visiting at Department of Pure Mathematics and Applied Statistics, University of Cambridge, Cambridge, England.

‡ Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

inner product $A * B$ is defined by

$$A(\mathbf{x}) * B(\mathbf{x}) = \sum_{(\mathbf{i})} a_{\mathbf{i}} b_{\mathbf{i}} g(\mathbf{i}) \mathbf{x}^{\mathbf{i}},$$

where $g(\mathbf{i}) = (1^{i_1} 2^{i_2} \cdots n^{i_n}) i_1! i_2! \cdots i_n!$:

(iii) $N(A(\mathbf{x})) = A(\mathbf{x})|_{\mathbf{x}=(1,1,\dots,1)}$.

Proof. See Read [4].

The use of Theorem 1 is illustrated by Example 1 of § 3.

Since h_n^m is a symmetric function, the computation of $H_m(n)$ may be simplified by expressing h_n^m as a linear combination of Schur functions and by using the orthogonality relation for Schur functions.

Let $\{\lambda\}$ be the Schur function associated with the partition (λ) of n. Then $h_n = \{n\}$ gives the representation of the cycle index polynomial for S_n in terms of Schur functions.

LEMMA 1. $h_n^m = \sum_{(\lambda)} \alpha_\lambda \{\lambda\}$, *where the summation is over all partitions of mn.*

Proof. h_n^m is a symmetric function.

LEMMA 2 (Orthogonality relation).

$$N(\{\lambda\} * \{\mu\}) = \begin{cases} 1 & \textit{if } (\lambda) \equiv (\mu), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. See Littlewood [5].

THEOREM 2. $H_m(n) = \Sigma_{(\lambda)} \alpha_\lambda^2$.

Proof. The proof is direct from Theorem 1 and Lemmas 1 and 2.

The remaining problem of expressing a symmetric polynomial as a linear combination of Schur functions may be carried out by a method given by Littlewood [5].

Let $\{\lambda\}$ be the Schur function corresponding to a partition (λ) of r. To evaluate $\{\lambda\}\{n\}$, construct the Young diagram for (λ) using "asterisks". Add to this diagram n "dots" in all possible ways, subject to the conditions

(i) resulting diagram is a Young diagram in the two symbols,

(ii) no two "dots" lie in the same vertical line.

The expansion of $\{\lambda\}\{n\}$ is the sum of the Schur functions corresponding to the partitions of $r + n$ generated in this manner. Example 2 of the following section illustrates this procedure.

3. Examples.

Example 1. Theorem 1 may be usefully employed for small values of n, for which the exponentiation of the polynomial h_n is readily constructed. We shall compute $H_m(2)$. Now $h_2 = \frac{1}{2}(x_1^2 + x_2)$, the cycle index polynomial for S_2. Therefore

$$\begin{aligned} H_m(2) = N(h_2^m * h_2^m) &= N\left(\frac{1}{2^m}(x_1^2 + x_2)^m * \frac{1}{2^m}(x_1^2 + x_2)^m\right) \\ &= \frac{1}{4^m} \sum_{\substack{i_1, i_2 \geqq 0 \\ i_1 + i_2 = m}} \binom{m}{i_1}^2 1^{2i_1} 2^{i_2} (2i_1)! (i_2)! \\ &= \frac{(m!)^2}{2^m} \sum_{i_1=0}^{m} \frac{2^{-i_1}}{(m - i_1)!} \binom{2i_1}{i_1}. \end{aligned}$$

This is in agreement with known results.

Example 2. We shall compute $H_3(2)$ using Theorem 2.

We shall represent $\{\mu\}$ by $[G(\mu)]$, where $G(\mu)$ is the Young diagram for (μ), and write $\{\mu\} \equiv [G(\mu)]$.

Then $\{2\} \equiv [*\,*]$, so

$$\{2\}^2 = [*\,*\cdot\cdot] + \left[\begin{matrix} * & * & \cdot \\ \cdot & & \end{matrix}\right] + \left[\begin{matrix} * & * \\ \cdot & \cdot \end{matrix}\right],$$

whence

$$\{2\}^2 = \{4\} + \{3, 1\} + \{2^2\}.$$

Finally

$$\begin{aligned}\{2\}^3 = \{2\}^2\{2\} &= \{4\}\{2\} + \{3, 1\}\{2\} + \{2^2\}\{2\} \\ &= \{6\} + 2\{5, 1\} + 3\{4, 2\} + 2\{3, 2, 1\} + \{4, 1^2\} + \{3^2\} + \{2^3\}\end{aligned}$$

Thus, from Theorem (2),

$$H_3(2) = N(\{2\}^3 * \{2\}^3) = 1^2 + 2^2 + 3^2 + 2^2 + 1^2 + 1^2 + 1^2 = 21.$$

4. Tabulation of $\{H_m(n)\}$, $\Phi_m(t)$. The generating function for $\{H_m(n)\}$ is given by

$$\Phi_m(t) = \sum_{n=0}^{\infty} t^n H_m(n) = \frac{f_m(t)}{(1-t)^{(m-1)^2+1}},$$

TABLE 1
Tabulation of $a_i^{(m)}$ for $m = 2, 3, 4, 5, 6$

m	2	3	4	5	6
i 0	1	1	1	1	1
1		1	14	103	694
2		1	87	4306	184015
3			148	63110	15902580
4			87	388615	567296265
5			14	1115068	9816969306
6			1	1575669	91422589980
7				1115068	490333468494
8				388615	1583419977390
9				63110	3166404385990
10				4306	3982599815746
11				103	3166404385990
12				1	1583419977390
13					490333468494
14					91422589980
15					9816969306
16					567296265
17					15902580
18					184015
19					694
20					1

where $f_m(t) = \sum_{i=0}^{(m-1)(m-2)} a_i^{(m)} t^i$ and all $a_i^{(m)}$ are integers.

Note that

$$f_m(t) = t^{(m-1)(m-2)} f_m(t^{-1})$$

is a consequence of

$$H_m(n) = (-1)^{m-1} H_m(-n-m).$$

The values of $a_i^{(m)}$ are given in Table 1, while Table 2 contains the initial segments of $\{H_m(n)\}$.

TABLE 2
Initial segments of $\{H_m(n)\}$ *for* $m = 2, 3, 4, 5, 6$

m	$H_m(n)$
2	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
3	1, 6, 21, 55, 120, 231, 406, 665, 1035, 1540, 2211
4	1, 24, 282, 2008, 10147, 40176, 132724, 381424, 981541, 2309384, 5045326
5	1, 120, 6210, 153040, 2224955, 22069251, 164176640, 976395820, 4855258305, 2085679285, 79315936751
6	1, 720, 202410, 20933840, 1047649905, 30767936616, 602351808741, 8575979362560, 94459713879600, 842286559093240, 6292583664553881

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