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THE ENUMERATION OF GENERALIZED DOUBLE STOCHASTIC NONNEGATIVE INTEGER SQUARE MATRICES*

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Abstract. The problem of enumerating generalized double stochastic integer square matrices is considered. The superposition theorem is used in conjunction with Schur functions to obtain the counting series for the 5 × 5 and 6 × 6 cases.

Key words. enumeration, stochastic matrices, generating functions, partitions, Schur functions, symmetric group

1. Introduction. Let $H_m(n)$ be the number of $m \times m$ matrices over $\mathbb{N} \equiv \{0, 1, 2, \cdots\}$ with line sum $n \in \mathbb{N}$. The purpose of this paper is to give the ordinary generating functions $\Phi_5(t)$, $\Phi_6(t)$ for the sequences $\{H_5(n)\}$ and $\{H_6(n)\}$ respectively with the anticipation that these additional sequences may be of service in determining the relationship among the $\Phi_i(t)$. The sequences for $\{H_3(n)\}$, $\{H_4(n)\}$ have been given by Sloane [1], together with references to the papers in which they first appear. The theorems supporting the computational method are given in § 2, while § 3 contains examples and a special case. The initial segments of sequences $\{H_i(n)\}$ and the generating functions $\Phi_i(t)$ are tabulated in § 4 for $i = 2, 3, 4$ (for completeness) and for $i = 5, 6$. Use has been made of Theorem 2, which is amenable to automatic computation.

2. Main theorem. The following result permits the computation of $H_m(n)$ by polynomial interpolation. The symmetry relation reduces the number of values of n at which $H_m(n)$ need be computed for interpolatory purposes.

CONJECTURE (Anand–Dumir–Gupta [2]).

(i) *$H_m(n)$ is a polynomial in n of degree* $(m-1)^2$.

(ii) $H_m(n) = (-1)^{m-1} H_m(-m-n)$.

Proof. See Stanley [3].

Values of $H_m(n)$ may be computed according to the following theorem.

THEOREM 1. $H_m(n) = N(h_n^m * h_n^m)$, *where*

(i) *h_n is the cycle index polynomial in the indeterminates $x_1 \cdots x_n$ for the symmetric group S_n;*

(ii) *if $A(\mathbf{x})$, $B(\mathbf{x})$ are two multivariate polynomials in $x_1 \cdots x_n$ such that*

$$A(\mathbf{x}) = \sum_{(\mathbf{i})} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}},$$

$$B(\mathbf{x}) = \sum_{(\mathbf{i})} b_{\mathbf{i}} \mathbf{x}^{\mathbf{i}},$$

where the summation is over all partitions $(\mathbf{i}) = 1^{i_1} 2^{i_2} \cdots n^{i_n}$ *of n, then the*

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inner product $A * B$ *is defined by*

$$A(\mathbf{x}) * B(\mathbf{x}) = \sum_{(\mathbf{i})} a_{\mathbf{i}} b_{\mathbf{i}} g(\mathbf{i}) \mathbf{x}^{\mathbf{i}},$$

where $g(\mathbf{i}) = (1^{i_1} 2^{i_2} \cdots n^{i_n}) i_1! i_2! \cdots i_n!$:

(iii) $N(A(\mathbf{x})) = A(\mathbf{x})|_{\mathbf{x}=(1,1,\dots,1)}$.

Proof. See Read [4].

The use of Theorem 1 is illustrated by Example 1 of § 3.

Since h_n^m is a symmetric function, the computation of $H_m(n)$ may be simplified by expressing h_n^m as a linear combination of Schur functions and by using the orthogonality relation for Schur functions.

Let $\{\lambda\}$ be the Schur function associated with the partition (λ) of n. Then $h_n = \{n\}$ gives the representation of the cycle index polynomial for S_n in terms of Schur functions.

LEMMA 1. $h_n^m = \sum_{(\lambda)} \alpha_\lambda \{\lambda\}$, *where the summation is over all partitions of* mn.

Proof. h_n^m is a symmetric function.

LEMMA 2 (Orthogonality relation).

$$N(\{\lambda\} * \{\mu\}) = \begin{cases} 1 & \text{if } (\lambda) \equiv (\mu), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. See Littlewood [5].

THEOREM 2. $H_m(n) = \Sigma_{(\lambda)} \alpha_\lambda^2$.

Proof. The proof is direct from Theorem 1 and Lemmas 1 and 2.

The remaining problem of expressing a symmetric polynomial as a linear combination of Schur functions may be carried out by a method given by Littlewood [5].

Let $\{\lambda\}$ be the Schur function corresponding to a partition (λ) of r. To evaluate $\{\lambda\}\{n\}$, construct the Young diagram for (λ) using "asterisks". Add to this diagram n "dots" in all possible ways, subject to the conditions

(i) resulting diagram is a Young diagram in the two symbols,

(ii) no two "dots" lie in the same vertical line.

The expansion of $\{\lambda\}\{n\}$ is the sum of the Schur functions corresponding to the partitions of $r + n$ generated in this manner. Example 2 of the following section illustrates this procedure.

3. Examples.

Example 1. Theorem 1 may be usefully employed for small values of n, for which the exponentiation of the polynomial h_n is readily constructed. We shall compute $H_m(2)$. Now $h_2 = \frac{1}{2}(x_1^2 + x_2)$, the cycle index polynomial for S_2. Therefore

$$\begin{aligned} H_m(2) = N(h_2^m * h_2^m) &= N\left(\frac{1}{2^m}(x_1^2 + x_2)^m * \frac{1}{2^m}(x_1^2 + x_2)^m\right) \\ &= \frac{1}{4^m} \sum_{\substack{i_1, i_2 \geqq 0 \\ i_1 + i_2 = m}} \binom{m}{i_1}^2 1^{2i_1} 2^{i_2} (2i_1)!(i_2)! \\ &= \frac{(m!)^2}{2^m} \sum_{i_1=0}^{m} \frac{2^{-i_1}}{(m-i_1)!} \binom{2i_1}{i_1}. \end{aligned}$$

This is in agreement with known results.

Example 2. We shall compute $H_3(2)$ using Theorem 2.

We shall represent $\{\mu\}$ by $[G(\mu)]$, where $G(\mu)$ is the Young diagram for (μ), and write $\{\mu\} \equiv [G(\mu)]$.

Then $\{2\} \equiv [*\,*]$, so

$$\{2\}^2 = [*\,*\cdot\cdot] + \left[\begin{smallmatrix}*&*&\cdot\\ \cdot&&\end{smallmatrix}\right] + \left[\begin{smallmatrix}*&*\\ \cdot&\cdot\end{smallmatrix}\right],$$

whence

$$\{2\}^2 = \{4\} + \{3,1\} + \{2^2\}.$$

Finally

$$\begin{aligned}\{2\}^3 &= \{2\}^2\{2\} = \{4\}\{2\} + \{3,1\}\{2\} + \{2^2\}\{2\}\\ &= \{6\} + 2\{5,1\} + 3\{4,2\} + 2\{3,2,1\} + \{4,1^2\} + \{3^2\} + \{2^3\}\end{aligned}$$

Thus, from Theorem (2),

$$H_3(2) = N(\{2\}^3 * \{2\}^3) = 1^2 + 2^2 + 3^2 + 2^2 + 1^2 + 1^2 + 1^2 = 21.$$

4. Tabulation of $\{H_m(n)\}$, $\Phi_m(t)$. The generating function for $\{H_m(n)\}$ is given by

$$\Phi_m(t) = \sum_{n=0}^{\infty} t^n H_m(n) = \frac{f_m(t)}{(1-t)^{(m-1)^2+1}},$$

TABLE 1

Tabulation of $a_i^{(m)}$ for $m = 2, 3, 4, 5, 6$

m	2	3	4	5	6
i 0	1	1	1	1	1
1		1	14	103	694
2		1	87	4306	184015
3			148	63110	15902580
4			87	388615	567296265
5			14	1115068	9816969306
6			1	1575669	91422589980
7				1115068	490333468494
8				388615	1583419977390
9				63110	3166404385990
10				4306	3982599815746
11				103	3166404385990
12				1	1583419977390
13					490333468494
14					91422589980
15					9816969306
16					567296265
17					15902580
18					184015
19					694
20					1

where $f_m(t) = \sum_{i=0}^{(m-1)(m-2)} a_i^{(m)} t^i$ and all $a_i^{(m)}$ are integers.

Note that

$$f_m(t) = t^{(m-1)(m-2)} f_m(t^{-1})$$

is a consequence of

$$H_m(n) = (-1)^{m-1} H_m(-n-m).$$

The values of $a_i^{(m)}$ are given in Table 1, while Table 2 contains the initial segments of $\{H_m(n)\}$.

TABLE 2

Initial segments of $\{H_m(n)\}$ *for* $m = 2, 3, 4, 5, 6$

m	$H_m(n)$
2	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
3	1, 6, 21, 55, 120, 231, 406, 665, 1035, 1540, 2211
4	1, 24, 282, 2008, 10147, 40176, 132724, 381424, 981541, 2309384, 5045326
5	1, 120, 6210, 153040, 2224955, 22069251, 164176640, 976395820, 4855258305, 2085679285, 79315936751
6	1, 720, 202410, 20933840, 1047649905, 30767936616, 602351808741, 8575979362560, 94459713879600, 842286559093240, 6292583664553881

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