Lower Bounds on Lotto Designs

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Abstract

An $LD(n, k, p, t; b)$ Lotto Design is a set of b k-sets (blocks) of an n-set such that any p-set intersects at least one k-set in t or more elements. Let $L(n, k, p, t)$ denote the minimum number of blocks in any $LD(n, k, p, t; b)$ Lotto Design. We introduce several lower bounds for $L(n, k, p, t)$ including a generalization of the Schönheim bound for covering designs.

1 Introduction

There are many lotteries in the world. They tend to be very similar in operation. The way most lotteries work is as follows. For a small fee, a person chooses k numbers from n numbers or has the numbers chosen randomly for him. This is the ticket. At a certain point no more tickets are sold and the government or casino picks p numbers from the n numbers, usually in a random fashion. These are called the winning numbers. If any of the tickets sold match t of the winning numbers then a prize is given to the holder of the matching ticket. The larger the value of t, the larger the prize. Usually t must be three or more to receive anything. Many people and researchers are interested to know what is the minimum number of tickets necessary to ensure that at least one ticket will intersect the winning numbers in t or more numbers. For an enlightening overview on this subject see Colbourn and Dinitz[5].

More formally we may define an (n, k, p, t) Lotto Design to be a set of k-sets (blocks or tickets) of an n-set such that any p-set intersects at least one k-set in t or more elements. An LD$(n, k, p, t; b)$ is a (n, k, p, t) Lotto Design with exactly b blocks. Let $L(n, k, p, t)$ denote the minimum number of blocks in any $LD(n, k, p, t; b)$. Clearly these definitions mirror the lottery situation and the fundamental problem becomes to find or estimate $L(n, k, p, t)$. The following is a typical example of a Lottery Design.

Example 1.1 : $LD(13, 6, 5, 3; 5) = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 6, 7\}, \{1, 2, 3, 5, 6, 7\}, \{8, 9, 10, 11, 12, 13\}\}$

Let f_i denote the number of elements of frequency i and let f_i^+ denote the number of elements of frequency i or more. Thus in the previous example $f_0 = 0, f_1 = 6, f_2 = 0, f_3 = 1, f_4 = 6$ and $f_0^+ = 13, f_1^+ = 13, f_2^+ = 7, f_3^+ = 7, f_4^+ = 6, f_5^+ = 0$
.

The main goal of this paper is to state and prove several lower bounds for $L(n, k, p, t)$. Two special subclasses of Lotto designs are covering and Turán designs. A (n, k, t) covering design is an (n, k, k, t) Lotto design and a (n, p, t) Turán design is an (n, t, p, t) Lotto design. Both covering and Turán designs have been thoroughly studied. A good introduction to covering designs and Turán designs may be found in [5]. A well known lower bound for an (n, k, t) covering designs was formulated by Schönheim [9] which, using our terminology, is:

$$L(n, k, k, t) = \left\lceil \frac{n}{k} L(n-1, k-1, k-1, t-1) \right\rceil . \qquad (1)$$

As covering and Turán designs are complements of each other, see Bate [1], a version of (1) may be restated for Turán designs as follows:

$$L(n,t,p,t) = \left\lceil \frac{n}{n-t} L(n-1,t,p,t) \right\rceil . \qquad (2)$$

More generally Bate[1] proved that the complement of an (n, k, p, t) Lotto design is an $(n, n-k, n-p, n-k-p+t)$ Lotto design.

2 History

Most known lower bounds for Lotto Designs deal with the case $t = 2$. One of the earliest mathematical studies on Lotto Designs was performed by Bate in his Doctoral thesis [1]. However, Bate did not use the term Lotto Designs. Instead, he called them "generalized (T, K, L, V) designs". In his work, the emphasis was in dealing with designs where the value of t was 2. By employing a graph theoretical approach, Bate showed the following:

Theorem 2.1 *:*

$$L(v,2,l,2) = \left\lceil \frac{v^2-(l-1)v}{2(l-1)} \right\rceil .$$

This particular result was first proved by Turàn [10] in 1941 and since then, additional proofs have appeared.

A formula for $L(v, 3, 3, 2)$ was also given in Bate's thesis. This result was independently derived by Brouwer [3] using Covering designs and stated in terms of Covering numbers. However, we will state Bate's version.

Theorem 2.2 *:*

$$L(v,3,3,2) = \begin{cases} \lceil \frac{v^2-2v}{12} \rceil & \textit{if } v \equiv 2,4,6 \textit{ (mod 12))} \\ \lceil \frac{v^2-2v}{12} \rceil + 1 & \textit{if } v \equiv 0,8,10 \textit{ (mod 12))} \\ \lceil \frac{v^2-v}{12} \rceil & \textit{if } v \equiv 1,3,5,7 \textit{ (mod 12))} \\ \lceil \frac{v^2-v}{12} \rceil + 1 & \textit{if } v \equiv 9,11 \textit{ (mod 12))} \end{cases}$$

In addition to these results, Bate proved good lower bounds for $L(v, 4, 4, 2)$, $L(v, 3, 4, 2)$, $L(v, 3, 5, 2)$, $L(v, 4, 3, 2)$ and $L(v, 4, 5, 2)$.

Füredi, Székely and Zubor [7] determined the following lower bound for $L(n, k, p, 2)$:

$$L(n,k,p,2) \geq_{a_1+...+a_{p-1}=n}^{min} \frac{1}{k}\left(a1\left\lceil\frac{a_1-1}{k-1}\right\rceil + ... + a_{p-1}\left\lceil\frac{a_{p-1}-1}{k-1}\right\rceil\right).$$

They also gave a non-exhaustive list of cases where the bounds meet. The approach taken by the authors generated good lower bounds for $L(n, k, p, 2)$. However, no known similar technique can be applied to generate good lower bounds for $L(n, k, p, t)$ where $t > 3$.

Bate and van Rees[2] determined the values for $L(n, 6, 6, 2)$ for $n \leq 54$ which are stated in the following theorem.

Theorem 2.3 *:*

n	*k*	*p*	*t*	*L(n,k,p,t)*
35	6	6	2	9
36	6	6	2	9
37	6	6	2	9
38	6	6	2	11
39	6	6	2	11
40	6	6	2	12
41	6	6	2	13
42	6	6	2	13
43	6	6	2	14
44	6	6	2	15
45	6	6	2	15
46	6	6	2	16
47	6	6	2	17
48	6	6	2	18
49	6	6	2	19
50	6	6	2	19
51	6	6	2	20
52	6	6	2	21
53	6	6	2	22
54	6	6	2	23

The following lower bound for $L(n,k,p,t)$ made be found in [8].

Theorem 2.4 *:*

$$L(n,k,p,t) \geq \frac{\binom{n}{k}}{\sum_{i=t}^{min(k,p)} \binom{k}{i}\binom{n-k}{p-i}}.$$

De Caen [6] determined a lower bound for Turán designs which states:

Theorem 2.5 *:*

$$T(n,p,t) \geq \frac{\binom{n}{t}}{\binom{p-1}{t-1}} \frac{n-p+1}{n-t+1}.$$

Both Bate [1] and Brouwer and Voorhoeve [4] determined that

$$L(n,k,p,t) \geq \frac{T(n,p,t)}{\binom{k}{t}}.$$

This result along with that of de Caen yield the following general lower bound formula for $L(n,k,p,t)$.

Theorem 2.6 *:*

$$L(n,k,p,t) \geq \frac{\binom{n}{t}}{\binom{p-1}{t-1}\binom{k}{t}} \frac{n-p+1}{n-t+1}.$$

So for $t = 2$ there are good bounds for Lotto Designs and for larger t we have bounds that give a starting point to our search for new bounds which we do in the next section.

3 Schönheim's Lower Bound

In this section we shall generalize Schönheim's lower bound formula for covering designs, as stated by (1). But first we need to set up some notation and definitions.
We shall denote an (n, k, p, t) Lotto design design by the pair $(X, \mathcal{B})$ where X is the set of elements in the design and $\mathcal{B}$ are the blocks of the design. Unless otherwise stated, the set X will contain the elements 1,2, ..., n. We let $X(n)$ denote the set $\{1, 2, ..., n\}$.

Definition 3.1 *: If $(X, \mathcal{B})$ is an (n, k, p, t) Lotto design and S is a subset of X, let $\mathcal{B}^{S*} = \{T \in \mathcal{B} : |S \cap T| \leq k - t\}$ and let $\mathcal{B}^S = \{T^* :$ if $T \in \mathcal{B}^{S*}$ then $T^* = T \setminus S \cup X^T$ where $|X^T| = |T \cap S|$ and $X^T \subseteq X \setminus S\}$.*

First we investigate what happens when some elements are deleted from a Lotto design and the blocks whose size is smaller than t are thrown away and the blocks whose size is equal to or larger than t are filled up with remaining elements still in the design.

Theorem 3.1 *: Let $(X, \mathcal{B})$ be an (n, k, p, t) Lotto design and let S be a subset of X such that $|S| \leq n - p$ and $|S| \leq n - k$. Then $(X \setminus S, \mathcal{B}^S)$ is an $(n - |S|, k, p, t)$ Lotto design.*

Proof : It is easy to see that blocks in $\mathcal{B}^S$ are made up of elements from $X \setminus S$ and since $n \geq k + |S|$, each block will not have repeated elements (although there could be repeated blocks). Let P be a p-set from $X \setminus S$, then since $(X, \mathcal{B})$ is a Lotto design, there exists some $T \in \mathcal{B}$ such that $|T \cap P| \geq t$. Since $P \cap S = \emptyset$, then $k \geq |(P \cap T)| + |(S \cap T)| \geq t + |S \cap T|$. Hence $|S \cap T| \leq k - t$. As $\mathcal{B} \neq \emptyset$, $T \in \mathcal{B}^{S*}$ and we are done. □

Corollary 3.1 *: If an (n, k, p, t) Lotto design with x blocks exist with $n \geq p + k - t + 1$ and $n \geq 2k - t + 1$ then an $(n - k + t - 1, k, p, t)$ Lotto design with y blocks exists where $y < x$*

Proof : Consider a set $S \subseteq X$ where $|S| = k - t + 1$. Then $|S| \leq n - p$ and $|S| \leq n - k$ and hence Theorem 3.1 holds which immediately give the desired result. □

Example 3.1 *:* Consider the following 20 blocks of an $(14, 5, 4, 3)$ Lotto Design $(X, \mathcal{B})$ where $X = \{0, 1, ..., 13\}$ which we obtained by simulated annealing :
$\{0, 1, 2, 5, 7\}$, $\{1, 3, 4, 6, 7\}$, $\{2, 3, 5, 8, 10\}$, $\{0, 3, 6, 9, 10\}$, $\{2, 4, 6, 9, 11\}$, $\{1, 2, 4, 10, 11\}$, $\{5, 6, 7, 10, 11\}$, $\{0, 1, 6, 8, 12\}$, $\{4, 5, 6, 8, 12\}$, $\{0, 4, 8, 9, 12\}$, $\{5, 7, 9, 10, 12\}$, $\{1, 2, 3, 11, 12\}$, $\{0, 7, 8, 11, 12\}$, $\{0, 3, 4, 5, 13\}$, $\{2, 7, 8, 9, 13\}$, $\{1, 5, 9, 11, 13\}$, $\{3, 8, 9, 11, 13\}$, $\{1, 8, 10, 11, 13\}$, $\{0, 2, 6, 12, 13\}$, $\{4, 7, 10, 12, 13\}$

Let $S = \{6, 8, 11, 12\}$. By theorem 3.1, the following collection of 17 blocks form an $(10, 5, 4, 3)$ Lotto design with 17 blocks and whose elements are from $\{0, 1, 2, 3, 4, 5, 7, 9, \ 10 \ , 13\}$.
$\{0, 1, 2, 5, 7\}$, $\{1, 3, 4, 5, 7\}$, $\{2, 3, 5, 9, 10\}$, $\{0, 3, 5, 9, 10\}$, $\{2, 4, 5, 9, 13\}$, $\{1, 2, 4, 10, 13\}$, $\{4, 5, 7, 10, 13\}$, $\{0, 4, 5, 9, 10\}$, $\{5, 7, 9, 10, 13\}$, $\{1, 2, 3, 4, 5\}$, $\{0, 3, 4, 5, 13\}$, $\{2, 5, 7, 9, 13\}$, $\{1, 5, 9, 10, 13\}$, $\{3, 4, 9, 10, 13\}$, $\{1, 2, 9, 10, 13\}$, $\{0, 2, 3, 7, 13\}$, $\{4, 7, 9, 10, 13\}$.

We define the next term so that we can evaluate y in corollary 3.1.

Definition 3.2 *: If $(X, \mathcal{B})$ is an (n, k, p, t) Lotto design and S is a subset of X, then we define $deg_{\mathcal{B}}(S) = |\{T \in \mathcal{B} : S \subseteq T\}|$.*

Lemma 3.1 *: Let $(X, \mathcal{B})$ be an (n, k, p, t) Lotto design and $S \subseteq X$ such that $|S| = k - t + 1$. Then $|\mathcal{B}^S| = |\mathcal{B}| - deg_{\mathcal{B}}(S)$.*

Proof : We see that $|\mathcal{B}^S| = |\{T \in \mathcal{B} : |S \cap T| \leq k - t\}|$ and $deg_{\mathcal{B}}(S) = |\{T \in \mathcal{B} : |S \cap T| = k - t + 1\}|$. It is easy to see that any $T \in \mathcal{B}$ must be counted in one and only one of the two stated collections. Thus $|\mathcal{B}| = |\mathcal{B}^S| + deg_{\mathcal{B}}(S)$. That is, $|\mathcal{B}^S| = |\mathcal{B}| - deg_{\mathcal{B}}(S)$. $\square$

The following lemma implies that for a given (n, k, p, t) Lotto design $(X, \mathcal{B})$, any S with $|S| = k-t+1$, $n \geq 2k-t+1$ and $n \geq k-t+1+p$, will generate an $(n - k + t - 1, k, p, t)$ Lotto design. Notice that this is not necessarily true in general as stated by the formulation of Theorem 3.1. More specifically, Theorem 3.1 states that $\mathcal{B}^S$ must be non-empty. The following lemma states that this is always true under certain conditions.

Lemma 3.2 *: If $n \geq k-t+1+p$ and $n \geq 2k-t+1$ and $(X, \mathcal{B})$ is an (n, k, p, t) design then there cannot exist an $S \subseteq X$ where $|S| = k-t+1$ such that $S \subseteq T$ for all $T \in \mathcal{B}$.*

Proof: Suppose there is some (n,k,p,t) design $(X,\mathcal{B})$ and some $S \subseteq X$ where $|S| = k-t+1$ and $S \subseteq T$ for every $T \subseteq \mathcal{B}$. Then consider a p-set Q which has empty intersection with S. Such p-sets exists, since $n \geq k-t+1+p$. As $(X,\mathcal{B})$ is a Lotto design, there exists $T \in \mathcal{B}$ such that $|Q \cap T| \geq t$. However, as $Q \cap S = \emptyset$, $S \subseteq T$ and $|S| = k-t+1$, $|T \cap Q| \leq t-1$. This is a contradiction. Hence our lemma holds. □

We now state our main result which is used to derive a generalization of the Schönheim's lower bound for covering designs.

Theorem 3.2 *: If $(X,\mathcal{B})$ is an (n,k,p,t) Lotto design and $S \subseteq X$ such that $|S| = k-t+1$, $n \geq 2k-t+1$ and $n \geq k-t+1+p$ then*

$$|\mathcal{B}| \geq \left\lceil \frac{\binom{n}{k-t+1} L(n-k+t-1,k,p,t)}{\binom{n}{k-t+1} - \binom{k}{k-t+1}} \right\rceil$$

Proof : By Corollary 3.1 $|\mathcal{B}^S| \geq L(n-k+t-1,k,p,t)$ and by Lemma 3.1 $|B| - deg_{\mathcal{B}}(S) \geq L(n-k+t-1,k,p,t)$. By Lemma 3.2, sum over all $S \subseteq X$ such that $|S| = k-t+1$.

$$\sum |B| - \sum deg_{\mathcal{B}}(S) \geq \sum L(n-k+t-1,k,p,t).$$

By simple counting, we get the following :

$$\binom{n}{k-t+1}|B| - \binom{k}{k-t+1}|B| \geq \binom{n}{k-t+1} L(n-k+t-1,k,p,t).$$

By re-arranging the terms, we get the desired result. □

The following result follows immediate from Theorem 3.2.

Corollary 3.2 *:*

$$L(n,k,p,t) \geq \left\lceil \frac{\binom{n}{k-t+1} L(n-k+t-1,k,p,t)}{\binom{n}{k-t+1} - \binom{k}{k-t+1}} \right\rceil.$$

for $n \geq 2k-t+1$ and $n \geq k-t+1+p$.

We can simplify the previous result in the following two corollaries.

Corollary 3.3 : *If* $n \geq 2k - t + 1$ *and* $n \geq k - t + 1 + p$ *then* $L(n, k, p, t) > L(n - k + t - 1, k, p, t)$

Corollary 3.4 : *If* $n \geq 2k - t + 1$ *and* $n \geq k - t + 1 + p$ *then* $L(n, k, p, t) > L(m, k, p, t)$ *for all* $m \leq n - k + t - 1$.

Proof : Use Corollary 3.3 along with the fact that $L(n, p, k, t) \geq L(n - 1, k, p, t)$. □

We can now derive the Schönheim bound from Theorem 3.2.

Corollary 3.5 : $L(n', k', t', t') \geq \lceil \frac{n'}{k'} L(n' - 1, k' - 1, t' - 1, t' - 1) \rceil$.

Proof : Put $n = n'$, $k = n' - k'$, $t = n' - k'$ and $p = n' - t'$ in Theorem 3.2. We must show that $n \geq 2k - t + 1$ and $n \geq k - t + 1 + p$. We see that $2k - t + 1 = 2(n' - k') - (n' - k') + 1$ which is $n' - k' + 1 \leq n'$ as required. We also have $k - t + 1 + p = (n' - k') - (n' - k') + 1 + p = p + 1$. If $n = p$, then $n' = n' - t'$ and hence $t' = 0$ which is impossible. Thus $n \geq p + 1$ and both conditions of Theorem 3.2 are satisfied. Applying Theorem 3.2, we have

$$L(n, t, p, t) \geq \left\lceil \frac{n * L(n - 1, t, p, t)}{n - t} \right\rceil.$$

Since Turán designs and Covering designs are complements,

$$L(n, n-t, n-p, n-p) \geq \left\lceil \frac{n * L(n - 1, n - 1 - t, n - 1 - p, n - 1 - p)}{n - t} \right\rceil.$$

Hence

$$L(n, k', t', t') \geq \left\lceil \frac{n}{k'} L(n - 1, k' - 1, t' - 1, t' - 1) \right\rceil$$

which is the Schönheim lower bound for Covering designs. □

4 Other Lower Bound Formulas

In this section, we state several other lower bound results that we have discovered. In order to get improvement in the bounds we specialize the parameters of the Lotto Design. The first two lemmas deal with the case where p and t are close to each other.

Theorem 4.1 *:* $L(n,k,t+1,t) \geq min\{L(n,k,t-1,t-1), L(n-t+1, k-t+2, 2, 2)\}$.

Proof : If every $(t-1)$-set appears in some block of an $(n,k,t+1,t)$ Lotto design, then $L(n,k,t+1,t) \geq L(n,k,t-1,t-1)$. Otherwise, suppose the $(t-1)$-set $\{1,2,...,t-1\}$ does not appear in any block of an $(n,k,t+1,t)$ Lotto design, then consider the $(t+1)$-set $\{1,2,...,t-1,x,y\}$. It can only be represented by a k-set containing $\{x,y\}$ and exactly $t-2$ elements from $X(t)$. If all these $t-2$ elements were deleted, then an $L(n-t+1, k-t+2, 2, 2)$ would be obtained. Hence $L(n,k,t+1,t) \geq L(n-t+1, k-t+2, 2, 2)$. Combining both cases gives us the formula $L(n,k,t+1,t) \geq min\{L(n,k,t-1,t-1), L(n-t+1, k-1, 2, 2)\}$ as required. □

Theorem 4.2 *:* $L(n,k,t+2,t) \geq min\{L(n,k,t,t-1), L(n-t, k-t+2, 2, 2)\}$.

Proof : If every t-set intersects some block of an $(n,k,t+2,t)$ Lotto design in at least $t-1$ elements, then $L(n,k,t+2,t) \geq L(n,k,t,t-1)$. Otherwise, suppose that no $(t-1)$ elements of the t-set $\{1,2,...,t-1,t\}$ appear together in any block of an $(n,k,t+2,t)$ Lotto design. Consider the $(t+2)$-set $\{1,2,...,t-1,t,,x,y\}$. It can only be represented by a k-set that must include $\{x,y\}$, since at most $t-2$ elements from $\{1,2,3,...,t\}$ may appear together in a block of the design. Hence every pair $\{x,y\}$ where $x,y \notin X(t)$ must occur at least once in the $(n,k,t+1,t)$ Lotto design. This implies $L(n,k,t+2,t) \geq L(n-t, k-t+2, 2, 2)$. Combining both cases gives us the formula $L(n,k,t+2,t) \geq min\{L(n,k,t,t-1), L(n-t, k-t+2, 2, 2)\}$ as required. □

The following result gives the exact value of $L(n,k,p,t)$ for specific values for n, k, p and t.

Theorem 4.3 : *If* $n-k \geq p-t+1$ *and* $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$ *where* $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$ *then* $L(n,k,p,t) = r+1$.

Proof : Suppose $n-k \geq p-t+1$ and $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$ where $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$. We first need to show that an (n,k,p,t) Lotto design cannot have less than $r+1$ blocks. Suppose there is such a design with r blocks. As $n-k \geq p-t+1$, pick $p-t+1$ elements from the complement of each block and denote this set as P. There are at most $r*(p-t+1)$ distinct elements chosen. Since $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$, then $r*(p-t+1) \leq p$. If $|P| < p$, add any other distinct elements to P until $|P| = p$. The set P intersects the complement of each block in at least $p-t+1$ elements and hence will intersect each block of the design in at most $t-1$ elements. This contradicts the assumption that there is a Lotto design with r blocks. Hence we conclude that $L(n,k,p,t) \geq r+1$. Now we need to show that $L(n,k,p,t) \leq r+1$. We can do this by constructing such a design. Since $\left\lceil \frac{r}{r+1} n \right\rceil \leq k$, then $(r+1)(n-k) \leq n$. So pick $r+1$ $(n-k)$-sets by filling them with distinct elements. This implies that no element in these blocks is repeated. Since $r = \left\lfloor \frac{p}{p-t+1} \right\rfloor$ then $r+1 > \frac{p}{p-t+1}$ which implies $p < (r+1)(p-t+1) \leq (r+1)(n-k)$. This means that least one of the $(n-k)$-sets intersects any p-set in at most $p-t$ elements. This means that the p-set will intersect the complement of the $(n-k)$-set in at least t elements. The complements of the $(n-k)$-sets is the (n,k,p,t) Lotto design that we desire. Hence we have shown that $L(n,k,p,t) \leq r+1$. Since we have inequalities in both direction, we conclude that $L(n,k,p,t) = r+1$. □

We conclude this section with lower bound formulas for specific values of p and t. Before we begin we need to state a useful result from Bate and a generalization of that result. [1].

Lemma 4.1 : *If* $L(n,k,p,t) \geq \frac{n}{k}$ *then there exist a minimal design which contains every element.*

Lemma 4.1 generalizes to the following result whose proofs are nearly identical.

Lemma 4.2 : *If there exists an* $LD(n,k,p,t;b)$ *Lotto design where* $b \geq \frac{n}{k}$ *then there exist an* $LD(n,k,p,t;b)$ *Lotto design which contains every element.*

Theorem 4.4 *:* $L(n,k,6,3) \geq min\{L(n,k,4,2), L(n-4,k-1,2,2)\}$.

Proof : Either each 4-set intersects some block of an $(n,k,6,3)$ Lotto design in at least two elements or some 4-set intersects each block of the design in at most one element. If every 4-set intersects some block of the design in at least 2 elements then $L(n,k,5,3) \geq L(n,k,4,2)$. If a 4-set, say $\{1,2,3,4\}$ does not intersect any block of an $(n,k,6,3)$ design in more than one element, then the 6-set $\{1,2,3,4,x,y\}$ where $x,y \in X(n) \setminus \{1,2,3,4\}$ must be represented by a block that contains both x and y. This implies $L(n,k,6,3) \geq L(n-4,k-1,2,2)$. Combining both cases, we get the formula : $L(n,k,6,3) \geq min\{L(n,k,4,2), L(n-4,k-1,2,2)\}$. $\square$

Theorem 4.5 *: If* $k \geq 3$ *then* $L(2k+1,k,5,3) \geq 5$.

Proof : For $k = 3$, the design in question is a $(7,3,5,3)$ which is a Turán design. By complementing the design, we get a $(7,4,2,2)$ design. From Bate's tables, $L(7,4,2,2) = 5$, hence the result is true for $k = 3$. We will now prove the result for $k \geq 4$. Suppose there is an $(2k+1,k,5,3)$ Lotto design on four block. If a given 5-set is not represented in this design, then the 5-set will intersect every block of the complement in at least three elements. We will show that we can always construct such a 5-set that intersects every block of the complement in at least three elements. Let B_1, B_2, B_3, B_4 denote the four $(k+1)$-sets in the complement of the design. Suppose that there is an element of frequency zero in the design, say the element 1. Then that element appears in each block of the complement. From this point on, we will be speaking with respect to the complement blocks of the design unless otherwise stated. There are $4k$ spots left to be filled using $2k$ elements. The average frequency is 2.
Case 1 : If there is another element, say 2, of frequency 4, then there are $4k-4$ spots left to be filled by $2k-1$ elements. The average frequency of the remaining elements is $1 + \frac{2k-3}{2k-1}$. Since $k \geq 4$ then there exists an element of frequency 2 or higher. For if not, then every element has frequency at most 1 and the total number of spots that can be filled is $8 + 1(2k-1) = 2k+7 < 4k+4$, as $k \geq 4$. So suppose that the element 3 has frequency at least 2. Without loss of generality, assume the element 3 belong to B_1 and B_2. The 5-set $\{1,2,3,x,y\}$ where $x \in B_3$ and $y \in B_4$ is a 5-set that intersects each block in at

least three elements which imply that this 5-set is not represented by the Lotto design. This is a contradiction and hence it is impossible to have two elements of frequency 4 in the complement.

Case 2 : If there is an element, say 2, of frequency 3, then there are $4k-3$ spots left to be filled by $2k-1$ elements. The average frequency is $1+\frac{2k-2}{2k-1}$. If there is another element of frequency 3, say 3, then the blocks of the complement would look like: $B_1 = \{1, 2, , ...\}, B_2 = \{1, 2, 3, ...\}, B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, 3, ...\}$ or else $B_1 = \{1, 2, 3, ...\}, B_2 = \{1, 2, 3, ...\}, B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, ...\}$. If the former case, then the 5-set $\{1, 2, 3, x, y\}$ where $x \in B_1$ and $y \in B_4$ intersects each block of the complement in at least 3 elements which is a contradiction. Similarly if $B_1 = \{1, 2, 3, ...\}, B_2 = \{1, 2, 3, ...\}, B_3 = \{1, 2, 3, ...\}$ and $B_4 = \{1, ...\}$ then the 5-set $\{1, 2, 3, x, y\}$ where $x \in B_4$ and $y \in B_4$ intersects each block of the complement in at least 3 elements which is a contradiction. Hence there is only 1 element of frequency 3 in the complements which in turn implies there is exactly one element of frequency 1 in the original blocks. Let the element 2 occur in B_1, B_2 and B_3. Thus all other elements must have frequency 2, that is, there are $2k-2$ elements of frequency 2 in the complements of the blocks. There must be an element of frequency 2 in block B_4, say 3. Since it has frequency 2, it must appear in one of B_1, B_2 or B_3. Without loss of generality, assume that element 3 occurs in B_3 and B_4. Since B_3 has room for only $k-2$ additional elements, there is an element, say 4, that does not appear in B_3 but does appear in B_4. Without loss of generality, assume that $4 \in B_1$ and $4 \in B_4$. Now the 5-set $\{1, 2, 3, 4, x\}$ where $x \in B_1$ intersects every block of the complement in at least 3 elements. This is a contradiction and hence implies that there are no elements of frequency three in the complement, if there is an element of frequency four in the complement.

Case 3 : The remaining case is that all elements other than element 1 has frequency 2. Suppose element 2 belongs to both B_1 and B_2. There are $4k-2$ spots left to be filled by $2k-1$ elements. Since there are $2k$ spots remaining in blocks B_3 and B_4 together and only $2k-1$ elements remain, B_3 and B_4 shared a common element, say 3. Now, since B_1 and B_2 must be distinct blocks, suppose the element 4 is in B_1 and not in B_2. Since the element 4 has frequency 2, then suppose it also is in B_3. Finally there are $2k-2$ spots left to be filled between B_2 and B_4 using $2k-3$ elements. Hence, there exists a common element between these two blocks, say element 5. The 5-set $\{1, 2, 3, 4, 5\}$ intersects every block in the complement in at least three elements, which is a contradiction.

If there is an element of frequency zero in the Lotto design, every scenario yields a contradiction. Hence we conclude that there must not be any elements of frequency zero in the Lotto design. Now there must be at least one element of frequency 1 in the Lotto design. For if not, then every element in the Lotto design has frequency at least two, and hence the total in the design is at least $2(2k + 1) = 4k + 2 > 4k$ which is a contradiction. Thus, we can assume that element 1 has frequency 3 in the complement. Using the same reasoning again, there is another element of frequency 3, say the element 2. There are two cases for the arrangements of these two elements.

Case a : $B_1 = \{1, 2, ...\}$, $B_2 = \{1, 2, ...\}$, $B_3 = \{1, 2, ...\}$ and $B_4 = \{3, 4, 5, ..., k + 3\}$. There are $k - 2$ elements remaining to be used. These $k - 2$ elements cannot fill up the $k - 1$ spots in any one of the blocks B_1, B_2 or B_3. Hence there exists three elements, say 3,4 and 5 such that x is in B_1, y is in B_2 and z is in B_3 where $x, y, z \in \{3, 4, 5\}$. Now the 5-set $\{1, 2, 3, 4, 5\}$ intersects every block of the complement in at least three elements which is a contradiction.

Case b : $B_1 = \{1, 2, ...\}$, $B_2 = \{1, 2, ...\}$, $B_3 = \{1, ...\}$ and $B_4 = \{2, ...\}$. There are $2k$ spots remaining to be filled between blocks B_3 and B_4 using $2k - 1$ elements. So there exists some common element between these two blocks, say the element 3. If there is an element of frequency 3, say x that belongs to every block except say B_1, then the 5-set $\{1, 2, 3, x, y\}$ where $y \in B_1$ intersects every block of the complement in at least three elements. On the other hand, if no element of frequency three exist, then every remaining element must be of frequency two. Since B_1 and B_2 are distinct, let the element 4 be in B_1 and not in B_2. Then since the element 4 has frequency 2, then it must appear in either B_3 or B_4. Without loss of generality, assume it is in B_3. Now B_2 and B_4 have $2k - 2$ spots remaining to be filled between them using $2k - 3$ elements. This implies B_2 and B_4 must have a common element amongst them, say the element 5. Thus the 5-set $\{1, 2, 3, 4, 5\}$ intersects every block of the complement in at least 3 elements which is a contradiction.

Since every scenario leads to a contradiction, the assumption that there is an $(2k+1, k, 5, 3)$ Lotto design on 4 blocks must be false, and hence $L(2k+1, k, 5, 3) \geq 5$. $\square$.

Theorem 4.6 *: If* $k \geq 3$ *then* $(2k+2, k, 6, 3) \geq 4$.

Proof: Assume that $L(2k+2, k, 6, 3) \leq 3$ and consider an $(2k+2, k, 6, 3)$ Lotto design with three blocks. We shall construct a 6-set that is not represented by this design and hence such a design cannot exist.
There are $3k$ total occurrences in the design. We consider the cases for the number of elements of frequency 0.
Case 1 : If $f_0 = 0$, then since each element can have frequency at most 3, the number of element of f_i for $i \geq 2$ is $k-2$. This is because if there are $k-1$ or more elements of frequency at least 2, then there are at most $k+3$ elements of frequency 1 which give a total occurrences of at least $(k-1)2+(k+3) = 2k-2+k+3 = 2k+1 > 3k$. This implies that there are at least two elements of frequency one from each block. Pick two such elements from each block to form a 6-set that will not be represented.
Case 2 : If $f_0 = 1$, pick that element as part of our 6-set. Pick any element a of frequency 2 to be in the 6-set. Now pick 2 frequency 1 elements from two distinct blocks which occur with a in the design. These two elements must exist, for if not then there are at least k elements of frequency 2. This gives a total occurrence count of at least $2k+(k+2) > 3k$ which is not possible. Now consider the block which does not contain a. It has at least 2 elements of frequency 1 because there are at most $k-2$ elements of frequency two left. Pick them for our 6-set. The just formed 6-set is not represented in the design.
Case 3: If $f_0 = 2$, then pick these two elements as part of our 6-set. Since $k \geq 4$, there exists two elements of frequency 2, say $\{a, b\}$ that are in the same block, say B_1. Pick them to be in our 6-set. If there is a block that does not contain either of these two elements, then it must have at least 2 elements of frequency 1 as there are at most $k-2$ elements of frequency two left. Pick them for our 6-set. The just formed 6-set is not represented in the design. Otherwise, if $\{a, b\}$ intersects all three blocks, then B_2 and B_3 must each contain an element of frequency one. Pick them for our 6-set. The just formed 6-set is not represented in the design.

The cases where $3 \leq f_0 \leq 5$ trivially lead to contradictions. Hence $L(v, k, 6, 3) \geq 3.\Box$

Theorem 4.7 *: $L(3k+1, k, 7, 3) \geq 6$ for $k \geq 3$.*

Proof : Suppose there is an $(3k+1, k, 7, 3)$ Lotto design with five blocks.Let us denote these blocks as B_1, B_2, B_3, B_4 and B_5. By lemma 4.2, we may assume every element appears in this design. Looking at other frequencies, we see that at least $k+2$ elements have frequency one and at most $2k-1$ elements have frequency two or more. We will proceed to show that these five blocks cannot form a $(3k+1, k, 7, 3)$ Lotto design. We will do this be considering how many blocks the frequency one elements appear in. We begin be considering the case where the elements of frequency one appear in exactly two blocks.

Case 1 : If the elements of frequency one appear in only two blocks, say B_1 and B_2. Let a and b be two elements of frequency one from B_1 and let c and d be two elements of frequency one from B_2. There are $3k+1$ elements in the Lotto design and at most $2k$ of them are in B_1 and B_2. Thus at least $k+1$ elements must occur only in B_3, B_4 or B_5. It is easy to see that in these $k+1$ elements, there are at least three that have frequency two. Let x, y, z be three such elements. These three elements can occur in B_3, B_4 and B_5 in three different ways.
Case 1a : $B_3 = \{x, y, ...\}$, $B_4 = \{x, z, ...\}$ and $B_5 = \{y, z, ...\}$. Then the 7-set $\{a, b, c, d, x, y, z\}$ is not represented by the design.
Case 1b : $B_3 = \{x, y, z, ...\}$, $B_4 = \{x, y, ...\}$ and $B_5 = \{z, ...\}$. Then there are at least $k+1-3 = k-2$ other elements that occur only in B_3, B_4 or B_5. If they were all frequency three elements then they would occur in B_3, B_4 and B_5. But then B_3 would contain $k-2+3 = k+1$ elements which is a contradiction. So, one of the $k-2$ elements, say element w, has frequency two and so appears in B_4 and B_5. The 7-set $\{a, b, c, d, y, z, w\}$ is not represented in the design.
Case 1c : $B_3 = \{x, y, z, ...\}$ and $B_4 = \{x, y, z, ...\}$. Then there are at least $k+1-3 = k-2$ other elements that occur only in B_3, B_4 or B_5. Since $B_3 \neq B_4$, one of these $k-2$ elements, say element w, must have frequency two and appear in B_5. It also appears in either B_3 or B_4. In either case, it takes us back to case 1b.

We now consider the case where the elements of frequency one appear in three blocks.
Case 2 : Suppose B_2 contains exactly one element of frequency one, B_3 contain exactly one element of frequency one and B_1 contains the rest of the elements of frequency one (that is, B_4 and B_5 have no elements of frequency one). Since the number of elements of frequency one is at least $k+2$ and they must appear in B_1, B_2 or B_3, then B_1 must be made up entirely of frequency one elements. Let a and b be two elements of frequency one from B_1, and let c and d be the element of frequency one from B_2 and B_3 respectively. Since there are $2k-1$ elements remaining to be considered and only $2k-2$ spots opened between B_2 and B_3, there exists an element x such that $x \in B_4 \cap B_5$ and x is not in any other block. If B_2 and B_3 are disjoint then we must be able to find two elements $y \in B_2$ and $z \in B_3$ such that they do not appear together in B_4 and B_5. Then, the 7-set $\{a, b, c, d, x, y, z\}$ is not represented which is a contradiction. On the other hand, if there is an element e such that $e \in B_2 \cap B_3$, then there there exists another element $y \in B_4 \cap B_5$ such that y is not in any other blocks. Then the 7-set $\{a, b, c, d, e, x, y\}$ is not represented by any block in the design which is a contradiction.
Case 3 : Suppose the elements of frequency one appear at least twice in each of B_1 and B_2, exactly once in B_3 and do not appear in B_4 or B_5. Let a and b be two elements of frequency one from B_1, c and d be two elements of frequency one from B_2 and e the element of frequency one from B_3. Then there is at least one element x in both B_4 and B_5 such that it is not in any other block because the first three blocks contain at most $3k$ distinct elements. Also there must be an element y in B_3, B_4 or B_5 such that it is not in B_1 or B_2 since the first two blocks contain at most $2k$ distinct elements. The 7-set $\{a, b, c, d, e, x, y\}$ is not represented by any block of the design. This is a contradiction.
Case 4 : Suppose the frequency one elements appear at most $3k$ times and are contained within the blocks B_1, B_2 and B_3. Let a, b be frequency one elements in B_1, c, d be frequency one elements in B_2 and e, f be frequency one elements in B_3. There exists an element x not in B_1, B_2 or B_3 as there are $3k+1$ elements in the design and the first three blocks may contain at most $3k$ distinct elements. Since x has frequency greater than one but does not appear in B_1, B_2 or B_3 then x must appear in B_4 and B_5. The 7-set $\{a, b, c, d, e, f, x\}$ is not

represented by any block in the design. This is a contradiction.

We now deal with the case where the elements of frequency one appear in exactly four blocks.

Case 5 : Suppose B_1 has at least $k-1$ elements of frequency one and B_2, B_3, B_4 each have exactly one element of frequency one. Then all elements have frequency two or three except for possibly one element of frequency one. Let a, b be elements of frequency one chosen from B_1 and c, d, e be the elements of frequency one from B_2, B_3 and B_4 respectively. If every element between B_1, B_2 and B_3 are distinct, then there are elements $x \in B_2, y \in B_3$ of frequency at least two such that they do not appear together in any block. This is because there are $(k-1)^2$ $\{x, y\}$ pairs between B_2 and B_3. The number of these pairs that could occur together in B_4 and B_5 is $\begin{pmatrix} k \\ 2 \end{pmatrix} + \begin{pmatrix} k-1 \\ 2 \end{pmatrix}$. The 7-set $\{a, b, c, d, e, x, y\}$ is not represented by any block in the design which is a contradiction. On the other hand, if there is a element common between B_2 and B_3, say f, then we see that there are at most $2k-2$ distinct elements of frequency two to go into B_4 and B_5. This means that some element, say g, occurs in both B_4 and B_5. Then the 7-set $\{a, b, c, d, e, f, g\}$ is not represented by any block in the design.

Case 6 : Suppose that B_1 and B_2 each contain at least two elements of frequency one, B_3 and B_4 each contain exactly one element of frequency one. Let a, b be elements of frequency one from B_1, c, d be elements of frequency one from B_2, e be the element of frequency one from B_3 and f be the element of frequency one from B_4. Since B_1 and B_2 together may only contain $2k$ distinct elements, there is an element x different from e or from f that does not appear in B_1 or B_2. The 7-set $\{a, b, c, d, e, f, x\}$ is not represented by blocks of the design.

Case 7 : Suppose B_1, B_2, B_3 and B_4 each contain at least one element of frequency one and at least three of these blocks contain two or more elements of frequency one. It is trivial that a 7-set may be formed that is not represented by the design.

We now deal with the cases where the frequency one elements appear in every block.

Case 8 : If the elements of frequency one appear in each block exactly once, then $k = 3$,$f_1 = 5$, $f_2 = 5$ and $f_3^+ = 0$. Let a, b, c, d and e denote an element of frequency one from each block. Assume B_1

and B_2 contain an element x which has frequency two. Since at most five distinct elements appear in B_1 and B_2, there exists an element y not in B_1 or B_2 with frequency two. The 7-set $\{a, b, c, d, e, x, y\}$ is not represented by the blocks of the design.

Case 9 : Consider the case where B_1 contains at least two elements of frequency one and B_2, B_3, B_4 and B_5 each contain at least one element of frequency one. Let a, b be elements of frequency one chosen from B_1 and let c, d, e, f be elements of frequency one chosen from B_2, B_3, B_4 and B_5 respectively. Clearly there exists an element x that does not belong in B_1 and is different from a, b, c, d, e and f. The 7-set $\{a, b, c, d, e, f, x\}$ is a 7-set not represented by any block in the design.

We have considered every case possible and each case led to a contradiction. Hence our assumption that an $(3k + 1, k, 7, 3)$ Lotto design with five blocks exists cannot be correct, and thus $L(3k + 1, k, 7, 3) \geq 6$. □

Theorem 4.8 *: $L(k + 2, k, 4, 3) \geq 3$, for $k \geq 3$*

Proof : Suppose there is an $(k+2, k, 4, 3)$ Lotto design with 2 blocks. Since $\frac{k+2}{k} < 2$, by lemma 4.2 , we may assume that the Lotto design contains no element of frequency zero. We may conclude that there are 4 elements of frequency one and $k - 2$ elements of frequency 2. Since there are two blocks, the $k-2$ elements must appear once in each block. Each block must contain exactly two elements of frequency one. The elements of frequency one form a 4-set that is not represented. This contradicts the assumption that there is an $(k+2, k, 4, 3)$ Lotto design with 2 blocks. Thus we conclude that no design of size two exists, so $L(k + 2, k, 4, 3) \geq 3$. □

Theorem 4.9 *: $L(3k + 2, k, 8, 3) \geq 5$, for $k \geq 3$.*

Proof : Suppose there is a $(3k + 2, k, 8, 3)$ Lotto design on four blocks. Since $\frac{3k+2}{k} \leq 4$ for $k \geq 2$, by lemma 4.2, we may assume that this design has no elements of frequency zero. By analyzing the frequencies of the design, there are at least $2k+4$ elements of frequency one and at most $k-2$ elements of frequency two or higher. Each block must have at least two elements of frequency one. We can pick eight elements of frequency one, two from each block to form an 8-set that is not represented by the blocks of the design. Thus, there cannot exist an $(3k+2,k,8,3)$ Lotto design on 4 blocks. □

Theorem 4.10 : *If* $k \geq 4$, $L(2k+3,k,9,4) \geq 4$.

Proof : Suppose that $L(2k+3,k,9,4) = 3$, then there is an $(2k+3,k,9,4)$ Lotto design $\mathcal{B}$ with three blocks B_1, B_2 and B_3. Since $\frac{2k+3}{k} \leq 3$, lemma 4.2 implies that $\mathcal{B}$ has no elements of frequency zero. By analyzing the frequencies of elements in $\mathcal{B}$ we see that the number of elements of frequency one is at least $k+6$ and the number of elements of frequency two or higher is at most $k-3$. Since there are at most $k-3$ elements of frequency two or higher, each block must contain at least three elements of frequency one. From each block of the design, select three elements of frequency one. The nine selected elements form a 9-set that is not represented by the design. Hence $L(2k+3,k,9,4) \geq 4$. □

Theorem 4.11 : $L(3k+2,k,11,4) \geq 5$ *for* $k \geq 4$.

Proof : Suppose there exists an $(3k+2,k,11,4)$ Lotto design with four blocks B_1, B_2, B_3 and B_4. Since $\frac{3k+2}{k} \leq 4$, by lemma 4.2, we may assume that this design has no elements of frequency zero. By analyzing the frequencies of the design, there are at least $2k+4$ elements of frequency one and at most $k-2$ elements of frequency two or higher. This implies that each block of the design must contain at least two elements of frequency one. If there are at least three blocks with at least three elements of frequency one, then we may select three elements of frequency one from three of these blocks and two elements of frequency one from the remaining block in the design to form an unrepresented 11-set in the design. Otherwise there must two blocks with two elements of frequency one, say blocks B_3 and B_4. Then the blocks B_1 and B_2 must be made up of strictly frequency one elements since $f_0 \geq 2k+4$. Let x be an element that does not appear in B_1 or B_2. Then the 11-set consisting of three elements of frequency one from each of B_1 and B_2, two elements of frequency one from each of B_3 and B_4 and x form an unrepresented 11-set in the design. Thus the four blocks B_1, B_2, B_3 and B_4 cannot form an $(3k+2,k,11,4)$ Lotto design. This implies that $L(3k+2,k,11,4) \geq 5$. □

Theorem 4.12 : *If* $k \geq 4$, $L(3k+3,k,12,4) \geq 5$.

Proof: Suppose $L(3k+3, k, 12, 4) \leq 4$ and consider an $(3k+3, k, 12, 4)$ Lotto design with 4 blocks B_1, B_2, B_3 and B_4. As $3k + 3 \leq 4k$ for $k \geq 4$, lemma 4.2 implies that we may assume that our design has no elements of frequency zero. By analyzing the frequencies of the elements in the design, we conclude there are at least $2k + 6$ elements of frequency 1. We claim that each block in the design contains at least three elements of frequency 1. For if not, suppose B_4 has at most two elements of frequency 1. The remaining $k - 2$ elements in B_4 must have frequency 2 or more and hence must appear again in the other three blocks. Since there are at least $2k + 6 - 2 = 2k + 4$ elements of frequency 1 in B_1, B_2 and B_3, these three blocks may contain at most $3k-(2k+4) = k-4$ other elements which is less than the $k-2$ elements it must hold. Hence each block in the design has at least three elements of frequency 1. By selecting three elements of frequency 1 from each of the 4 blocks, we could form a 12-set that is not represented by any block in the design. Hence $L(3k + 3, k, 12, 4) \geq 5$. □

Theorem 4.13 : *If $k \geq 5$, $L(2k + 4, k, 12, 5) \geq 4$.*

Proof: Suppose $L(2k+4, k, 12, 5) \leq 4$ and consider an $(2k+4, k, 12, 5)$ Lotto design with 3 blocks B_1, B_2 and B_3. As $2k + 4 \leq 3k$ for $k \geq 5$, lemma 4.2 implies that we may assume that our design has no elements of frequency zero. By analyzing the frequencies of the elements in the design, we conclude there are at least $k + 8$ elements of frequency 1. We claim that each block in the design contains at least four elements of frequency 1. For if not, suppose B_3 has at most three elements of frequency 1. The remaining $k - 3$ elements in B_3 must have frequency 2 or more and hence must appear again in the other two blocks. Since there are at least $k + 8 - 3 = k + 5$ elements of frequency 1 in B_1 and B_2, these two blocks may contain at most $2k - (k + 5) = k - 5$ other elements which is less than the $k - 3$ elements it must hold. Hence each block in the design has at least four elements of frequency 1. By selecting four elements of frequency 1 from each of the 3 blocks, we could form a 12-set that is not represented by any block in the design. Hence $L(2k + 4, k, 12, 5) \geq 4$. □

5 Concluding Remarks

We have stated several lower bound formulas for $L(n, k, p, t)$. Results similar to those given by Theorem 4.4 through 4.13 may also be derived for other values of n, k, p and t. However, most lower bound formulas are not exact and hence, computing the value of $L(n, k, p, t)$ for every n, k, p and t is far from complete.

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