WHEN IS A PARTIAL LATIN SQUARE UNIQUELY COMPLETABLE, BUT NOT ITS COMPLETABLE PRODUCT?

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Abstract: Let P and Q be uniquely completable partial latin squares. It is an open problem to determine necessary and sufficient conditions so that the completable product $P \otimes Q$ is also uniquely completable. So far, only a few specific examples of P have been given such that the completable product of P with itself ($P \otimes P$) does *not* have a unique completion. In this paper we find a whole class of such partial latin squares.

Keywords: Latin square; partial Latin square; critical set in Latin square

1. Introduction

The problem of finding defining sets in finite combinatorial configurations has been considered in many branches of combinatorics. These branches include: block designs (see Street [23], Sarvate and Seberry [22]), graph colourings (see Mahmoodian and Mendelsohn [21]), latin squares (see Cooper, Donovan and Seberry, [8], Donovan and Howse [14]), F-squares (see Fatina, Seberry and Sarvate [16]) and Room squares (see Chaudhry and Seberry [7]. In this paper we focus on latin squares.

Gower conjectured that if P and Q are partial latin squares with unique completion, then the completable product $P \otimes Q$ will also have unique completion. This conjecture has been shown to be true in a number of cases:

- when either of P or Q has a strong completion; (Gower [17])
- when either of P or Q is not Bedford and Whitehouse totally weak (see Section 3 for definitions). (Bedford and Whitehouse [5])

However recently Adams and Khodkar have found a counterexample to this conjecture [2]. In this paper we give a whole class of counterexamples, based on the structure of the *array of alternatives* for a partial latin square. This work is of interest as it suggests that there is a fundamental difference between strong and weak critical sets and raises many new questions about critical sets and latin trades. For instance, it is well know that a critical set

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must intersect every latin trade in the underlying latin square. This result, and its proof, suggests that the underlying latin square, L, contains a partial latin square, (a "near" latin trade) which itself is not a latin trade but which gives rise to a latin trade in the direct product $L \times L$. One questions is, can we characterise these "near" latin trades?

2. Definitions

We start with basic definitions which allow us to state and prove our main results.

Let $N = N(n) = \{0, 1, 2, \ldots, n-1\}$. A *partial latin square* P of order n is a set of ordered triples of the form (i, j, k), where $i, j, k \in N$ with the following properties:

- if $(i, j, k) \in P$ and $(i, j, k') \in P$ then $k = k'$,
- if $(i, j, k) \in P$ and $(i, j', k) \in P$ then $j = j'$ and
- if $(i, j, k) \in P$ and $(i', j, k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set N such that if $(i, j, k) \in P$, the *entry* k occurs in cell (i, j). We sometimes refer to the entry k in cell (i, j) of a partial latin square P as $(i, j)_P$.

Let P be a partial latin square of order n and let a, b, c be distinct elements of $\{1, 2, 3\}$. The (a, b, c)-*conjugate* of entry $(x_1, x_2, x_3) \in P$ is defined to be (x_a, x_b, x_c) and the (a, b, c)-*conjugate* of P, written $P_{(a,b,c)}$, is defined to be

$$P_{(a,b,c)} = \{(x_a, x_b, x_c) \mid (x_1, x_2, x_3) \in P\}.$$

A partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N = N(n) = \{0, 1, 2, \ldots, n-1\}$ in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array.

For a given partial latin square P the set of cells $\mathcal{S}_P = \{(i, j) \mid (i, j, k) \in P,$ for some $k \in N\}$ is said to determine the *shape* of P and $|\mathcal{S}_P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. For each r, $1 \le r \le n$, let $\mathcal{R}_P^r$ denote the set of entries occurring in row r of P. Formally, $\mathcal{R}_P^r = \{k \mid (r, j, k) \in P\}$. Similarly, for each c, $1 \le c \le n$, we define $\mathcal{C}_P^c = \{k \mid (i, c, k) \in P\}$.

A partial latin square T of order n is said to be a *latin trade* (or *latin interchange*) if $T \neq \emptyset$ and there exists a partial latin square T' (called a *disjoint mate* of T) of order n, such that

- $\mathcal{S}_T = \mathcal{S}_{T'}$,
- if $(i, j, k) \in T$ and $(i, j, k') \in T'$, then $k \neq k'$,
- for each r, $1 \le r \le n$, $\mathcal{R}_T^r = \mathcal{R}_{T'}^r$, (the row r is *balanced*) and
- for each c, $1 \le c \le n$, $\mathcal{C}_T^c = \mathcal{C}_{T'}^c$ (the column c is *balanced*).

A partial latin square is said to be *uniquely completable* (UC) to a latin square L if L is the only latin square of order n which has element k in cell (i, j) for each $(i, j, k) \in P$.

If P and Q are partial latin squares of order m and n respectively, the *direct product* $P \times Q$ of order mn is the partial latin square with rows and columns indexed by $N(m) \times N(n)$ and shape

$$S_{P\times Q} = \{((i, j), (k, l)) \mid (i, k) \in \mathcal{S}_P \text{ and } (j, l) \in \mathcal{S}_Q\},$$

with cell $((i, j), (k, l)) \in \mathcal{S}_{P\times Q}$ containing entry $((i, k)_P, (j, l)_Q)$. The entry $((i, k)_P, (j, l)_Q)$ is sometimes represented by $(in + j, kn + l)$.

Next, let P and Q be partial latin squares of order m and n respectively. Let L and M be latin squares such that $P \subseteq L$ and $Q \subseteq M$. We define the partial latin square $P \otimes Q$ (the *completable product* of P with Q with respect to L and M) of order mn to be the partial latin square with rows/columns indexed by $N(m) \times N(n)$, having shape

$$S_{P \otimes Q} = \{((i,j),(k,l)) \mid (i,k) \in \mathcal{S}_P \text{ or } (j,l) \in \mathcal{S}_Q\},$$

with cell $((i,j),(k,l)) \in \mathcal{S}_{P \otimes Q}$ containing entry $((i,k)_L,(j,l)_M)$. If both P and Q are uniquely completable, then L and M are uniquely determined and we may omit the phrase "with respect to L and M" when defining the completable product. The next example illustrates this definition.

Example 1. Consider the partial latin squares P and Q, given below. It can be shown that P and Q are critical sets in the latin squares L and M respectively.

$P =$

0	

and $L =$

0	1
1	0

$Q =$

			4	3
	2			
		1		4
			0	2
3	4			

and $M =$

0	1	2	4	3
4	2	3	1	0
2	0	1	3	4
1	3	4	0	2
3	4	0	2	1

The completable product $P \otimes Q$ is a partial latin square contained in the latin square $L \otimes M$, as shown below.

$P \otimes Q =$

0	1	2	4	3				9	8
4	2	3	1	0		7			
2	0	1	3	4			6		9
1	3	4	0	2				5	7
3	4	0	2	1	8	9			
			9	8				4	3
	7					2			
		6		9			1		4
			5	7				0	2
8	9				3	4			

$L \otimes M =$

0	1	2	4	3	5	6	7	9	8
4	2	3	1	0	9	7	8	6	5
2	0	1	3	4	7	5	6	8	9
1	3	4	0	2	6	8	9	5	7
3	4	0	2	1	8	9	5	7	6
5	6	7	9	8	0	1	2	4	3
9	7	8	6	5	4	2	3	1	0
7	5	6	8	9	2	0	1	3	4
6	8	9	5	7	1	3	4	0	2
8	9	5	7	6	3	4	0	2	1

Observe that $P \otimes Q \subseteq L \times M$. Also, if L and M are latin squares, then $L \times M = L \otimes M$.

Let P be a partial latin square contained in the latin square L. If there exists a latin trade T in L such that $P \cap T = \emptyset$, then P is also contained in the latin square $(L \setminus T) \cup T'$, where T' is a disjoint mate of T. Therefore if P is a UC set in a latin square L, P must intersect every latin trade in L.

It is precisely this fact we make use of in Theorem 17 to show that in certain cases, a partial latin square P may be uniquely completable while the completable product $P \otimes P$ has more than one completion.

3. The array of alternatives

In this paper we gain insight into determining whether a partial latin square P is UC by studying its *array of alternatives* $\mathcal{A}_P$.

1. $\mathcal{A}_P$ is an $n \times n$ array with each cell containing either a subset of $N = \{0, 1, \dots, n-1\}$ or an asterisk $*$ (but not both);
2. If $(i,j,k) \in P$ for some entry k, the cell (i,j) in $\mathcal{A}_P$ contains an asterisk $*$;

3. If $(i,j) \notin S_P$ (i.e. the cell (i,j) is empty in P), the cell (i,j) in A_P contains the set $N \setminus (\mathcal{R}_P^i \cup \mathcal{C}_P^j)$ (i.e. the set of elements of N that appear in neither row i nor column j of P). (Note: it is possible that this set is empty, in which case P does not have completion to a latin square.)

An array of alternatives is a special type of a more general structure which we call an *array square*. An array square $\mathcal{A}$ of order n is an $n \times n$ array with rows and columns indexed by N with each cell containing either a subset of N or an asterisk $*$. Like partial latin squares, we may also consider an array square as a set of ordered triples. If $M \subseteq N$ or $M = *$ occurs in cell (i,j) of $\mathcal{A}$, then we say that $(i,j,M) \in \mathcal{A}$. Furthermore if $(i,j,M) \in \mathcal{A}$, we say that $(i,j)_{\mathcal{A}} = M$.

Let P be a partial latin square and A_P the corresponding array of alternatives. We say that the symbol $k' \in (i,j)_{A_P}$ is *forced out* of A_P if either:

(1) there exists $r > 0$ and $i_1, i_2, \ldots, i_r$ (all $\neq i$) with $k' \in (i_1,j)_{A_P} \cup \ldots \cup (i_r,j)_{A_P}$ and $|(i_1,j)_{A_P} \cup \ldots \cup (i_r,j)_{A_P}| = r$; or
(2) for some $a,b,c \in \{1,2,3\}$, where a,b,c are all distinct, the (a,b,c)-conjugate of (i,j,k'), in the corresponding array of alternatives for the $P_{(a,b,c)} = \{(x_a, x_b, a_c) \mid (x_1,x_2,x_3) \in P\}$, satisfies 1.

The reduced array of alternatives, RA_P, is the array obtained from A_P by successively removing symbols which are forced out until no more symbols can be forced out. Then the addition of an entry $(i,j;k)$ to P is said to be *semi-forced* if either:

1. k is the only symbol in $(i,j)_{RA_P}$; or
2. $k \in (i,j)_{RA_P}$ and k occurs exactly once in either the i^{th} row or j^{th} column of RA_P.

Then a UC set U is *near-strong* to the latin square L if we can find a sequence of sets $U = S_1 \subset \cdots \subset S_f = L$ such that each triple $t \in S_{v+1} \setminus S_v$ is semi-forced in S_v. For the purpose of this paper we call a UC set which is not near-strong *Bedford-Whitehouse totally weak* or just *totally weak*.

An open problem is the following.

> When is it possible that an array square $\mathcal{A}$ is the array of alternatives for some partial latin square P?

An important necessary condition for an array square to be the array of alternatives for some partial latin square P is the following.

Definition 2. Let $\mathcal{A}$ be an array square of order n. Then $\mathcal{A}$ is said to satisfy the *rectangular rule* if and only if for all $i,j,i',j',k \in N$, if $k \in (i,j)_{\mathcal{A}} \cap (i',j')_{\mathcal{A}}$, where $i \neq i'$ and $j \neq j'$, then

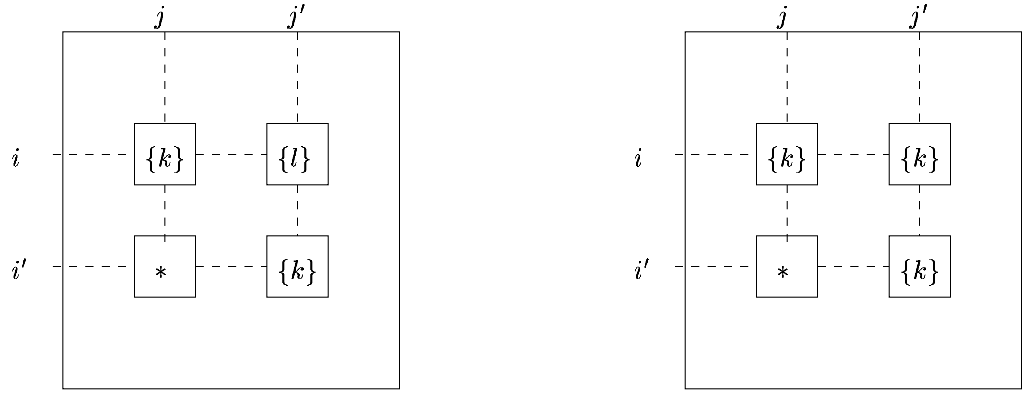
1. either $(i,j')_{\mathcal{A}} = *$ or $k \in (i,j')_{\mathcal{A}}$, and
2. either $(i',j)_{\mathcal{A}} = *$ or $k \in (i',j)_{\mathcal{A}}$.

The rectangular rule is illustrated in Figure 1.

Lemma 3. *Let $\mathcal{A}$ be an array square of order n. Then there exists a partial latin square P such that $\mathcal{A} = A_P$, the array of alternatives for P, only if $\mathcal{A}$ satisfies the rectangular rule.*

Proof. First suppose that $\mathcal{A}$ is an array square which does not follow the rectangular rule and that $\mathcal{A} = A_P$ is the array of alternatives for some partial latin square P. Then there exists i,j,i',j',k, such that $k \in (i,j)_{\mathcal{A}} \cap (i',j')_{\mathcal{A}}$, and (without loss of generality) $k \notin (i,j')_{\mathcal{A}} \neq *$. Since $k \in (i,j)_{\mathcal{A}} \cap (i',j')_{\mathcal{A}}$, the entry k occurs neither in row i nor column j' of P. Thus if $(i,j')_{\mathcal{A}} \neq *$, then $k \in (i,j')_{\mathcal{A}}$, a contradiction. □

Next we introduce some notation to link the idea of unique completability in a partial latin square to the "solvability" of an array square.



$\mathcal{A}$ $\qquad$ $\mathcal{B}$

Figure 1: The rectangular rule fails for $\mathcal{A}$, but not for $\mathcal{B}$.

Let the *shape* of an array square $\mathcal{A}$ be the set of cells

$$\{(i,j) \mid (i,j)_{\mathcal{A}} \neq *\}$$

and let the size of $\mathcal{A}$ be

$$|\{(i,j) \mid (i,j)_{\mathcal{A}} \neq *\}|.$$

We define the *join* ($\vee$) of two array squares $\mathcal{A}$ and $\mathcal{B}$ of the same shape to be the following:

$$\mathcal{A} \vee \mathcal{B} = \{(i, j, (i,j)_{\mathcal{A}} \cup (i,j)_B)\}.$$

That is, the join of $\mathcal{A}$ and $\mathcal{B}$ is formed by taking the union of the sets in corresponding cells in $\mathcal{A}$ and $\mathcal{B}$.

We can transform a partial latin square Q of order n into an array square $\mathcal{A}(Q)$ of order n as follows. If cell (i,j) is empty in Q, we place an asterisk in cell (i,j) of $\mathcal{A}(Q)$. Otherwise if cell (i,j) contains the entry k in Q, we place the set $\{k\}$ in cell (i,j) of $\mathcal{A}(Q)$.

We say that an array square $\mathcal{A}$ is *solvable* if there exists a partial latin square $Q \neq \emptyset$ (of the same size and shape as $\mathcal{A}$) such that $\mathcal{A} = \mathcal{A}(Q) \vee \mathcal{B}$ for some array square B.

An important fact to note for later: *a solvable array square may fail the rectangular rule.*

We say that an array square $\mathcal{A}$ is k-solvable if k is the maximum integer such that there exist distinct partial latin squares $Q_1, Q_2, \ldots, Q_k$ (each with the same size and shape as $\mathcal{A}$), with

$$\mathcal{A} = \mathcal{A}(Q_1) \vee \mathcal{B}_1 = \mathcal{A}(Q_2) \vee \mathcal{B}_2 = \cdots = \mathcal{A}(Q_k) \vee \mathcal{B}_k,$$

for some array squares $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$. (Note, that by distinct, we mean that some triples are different, not necessarily all.) If $k = 1$ we say that $\mathcal{A}$ is *uniquely solvable* (or *US*).

If A_P is the array of alternatives for some partial latin square P, then P has exactly k completions to a latin square if and only if A_P is k-solvable.

We say that an array square $\mathcal{A}$ is *flexible* if for all $(i,j) \in N \times N$,

1. either $(i,j)_{\mathcal{A}} = \{*\}$ or $|(i,j)_{\mathcal{A}}| \geq 2$, and
2. for all $k \in (i,j)_{\mathcal{A}}$, there exists $i' \neq i$ and $j' \neq j$ such that $k \in (i,j')_{\mathcal{A}}$ and $k \in (i',j)_{\mathcal{A}}$.

The following lemma has a straightforward proof.

Lemma 4. *If P is a totally weak UC partial latin square then A_P, the array of alternatives for P, is a US flexible array square.*

The next lemma is sneaky and useful in the proof of Lemma 6.

Lemma 5. *Let $\mathcal{A}$ be an array square of order n that is solvable and flexible. Suppose in addition that for every $(i,j) \in N \times N$, $|(i,j)_A| \le 2$. Then if there exists at least one ordered pair (i,j) such that $(i,j)_{\mathcal{A}} \ne *$, there exists a latin trade T with disjoint mate T' such that $\mathcal{A}(T) \vee \mathcal{A}(T') = \mathcal{A}$, and $\mathcal{A}$ is not uniquely solvable.*

Proof. Because $\mathcal{A}$ is flexible, we know that for all $(i,j) \in N \times N$, either $(i,j)_{\mathcal{A}} = *$ or $|(i,j)_{\mathcal{A}}| = 2$. Since $\mathcal{A}$ is solvable, let $T \ne \emptyset$ be a partial latin square and $\mathcal{B}$ an array square with the same shape and size as $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}(T) \vee \mathcal{B}$. Moreover, if $(i,j,k) \in T$, then let $(i,j)_{\mathcal{B}} = (i,j)_{\mathcal{A}} \setminus \{k\}$.

Let $(i,j,k) \in T$. Since $\mathcal{A}$ is flexible, there exists $j' \ne j$ such that $k \in (i,j')_{\mathcal{A}}$. We cannot have $(i,j',k) \in T$, because T is a partial latin square. Therefore $k \in (i,j')_{\mathcal{B}}$. But if row i has m non-asterisked cells in $\mathcal{A}$, then

$$\sum_j |(i,j)_{\mathcal{A}}| \le 2m,$$

where j sums over all values such that $(i,j)_{\mathcal{A}} \ne *$. Thus each non-asterisked cell in row i must have exactly two entries, and there is a unique column j' such that $k \in (i,j')_{\mathcal{B}}$.

Using a similar argument for the columns, there exists a partial latin square T' such that $\mathcal{B} = \mathcal{A}(T')$. Moreover T is a latin trade, with T' its disjoint mate. □

4. Constructing a latin trade

Lemma 6. *Let $\mathcal{A}$ be a flexible, uniquely solvable array square with exactly one cell containing a set of size 3, and every remaining cell containing either a set of size 2 or an asterisk. Let B be the unique partial latin square such that $\mathcal{A} = \mathcal{A}(B) \vee T$ for some array square T. Without loss of generality let $(0,0)_{\mathcal{A}} = \{0,1,2\}$ and let $(0,0,0) \in B$. Then there exists distinct rows $i, i' \ne 0$ and distinct columns $j, j' \ne 0$ such that $0 \in (i,0)_{\mathcal{A}}$, $0 \in (i',0)_{\mathcal{A}}$, $0 \in (0,j)_{\mathcal{A}}$ and $0 \in (0,j')_{\mathcal{A}}$.*

Proof. Since $\mathcal{A}$ is flexible, for each entry $k \in (i,j)_{\mathcal{A}}$, we must have $k \in (i,j')_{\mathcal{A}}$ and $k \in (i',j)_{\mathcal{A}}$ for some $i' \ne i$ and $j' \ne j$. By elementary counting arguments, when $i \ne 0$ and $j \ne 0$, i' and j' are unique. Also, for row 0, there exists exactly one entry k_1 such that $k_1 \in (0,j)_{\mathcal{A}} \cap (0,j')_{\mathcal{A}} \cap (0,j'')_{\mathcal{A}}$ for some distinct columns j, j' and j''. Similarly, for column 0, there exists exactly one entry k_2 such that $k_2 \in (i,0)_{\mathcal{A}} \cap (i',0)_{\mathcal{A}} \cap (i'',0,)_{\mathcal{A}}$ for some distinct rows i, i' and i''.

We consider a number of cases. First, suppose that $k_1 \notin \{0,1,2\}$. If $k_1 \in (r,0)_{\mathcal{A}}$ for some $r \ne 0$, then by the rectangular rule, $k_1 \in (0,0)_{\mathcal{A}}$, a contradiction. Thus in total k_1 must occur in an even number of non-asterisked cells, since each entry appears twice in a column ($\ne 0$) if at all. But as k_1 appears an odd number of times in row 0, it must then occur an odd number of times in some row $r \ne 0$. However previously we have observed that only row 0 has an entry occurring an odd number of times.

So k_1 is equal to either 0, 1 or 2. Suppose that $k_1 = 1$. Then k_1 must also occur three times in column 0, otherwise we have a contradiction as in the previous paragraph. Let $\mathcal{A}'$ be the array square formed by taking $\mathcal{A}$ and removing the entry 1 from the set $\{0,1,2\}$ in the cell $(0,0)$. Now $\mathcal{A}'$ does not obey the rectangular rule; however it remains both uniquely solvable and flexible. Also, every cell contains either an asterisk or a set of size 2. Thus from Lemma 5, $\mathcal{A}'$ is not uniquely solvable, a contradiction.

Similarly, $k_1 \ne 2$, so we must have $k_1 = 0$, and by the transpose argument $k_2 = 0$. □

In what follows, let P be a totally weak, uniquely completable partial latin square, with an array of alternatives A_P having the following properties.

1. Exactly one cell has a set of size 3.

2. Every remaining cell contains either a set of size 2 or an asterisk.

Eventually we will show that the completable product $P \otimes P$ is not uniquely completable by constructing a latin trade in $(L \times L) \setminus (P \otimes P)$, where L is the unique completion of P. First we will prove some elementary properties about P and $\mathcal{A}_P$.

Without loss of generality, let: $(0,0)_{\mathcal{A}_P} = \{0,1,2\}$, and $(0,0,0) \in L$.

Lemma 7. *There exist distinct entries $e_1, e_2, e_3, \ldots, e_z$, $z \geq 2$, such that:*

1. *$e_1 = 1$, $e_z = 0$, and $e_d \neq 2$, for $1 < d < z$, and*
2. *for each $1 \leq d \leq z-1$, $\{e_d, e_{d+1}\} = (0, j_d)_{A_P}$ for some $j_d \neq 0$, with $(0, j_d, e_d) \in L$.*

Proof. Since P is totally weak and uniquely completable, its array of alternatives A_P is flexible and uniquely solvable. So from the previous lemma, entry 0 occurs three times in row 0 and three times in column 0 of A_P. A simple counting argument shows that every other element that occurs in row/column 0 occurs exactly twice.

Now, $1 \in (0,0)_{A_P}$, so there must be a unique column $j_1 \neq 0$ such that $1 \in (0, j_1)_{A_P}$. Let $\{e_2\} = (0, j_1)_{A_P} \setminus \{1\}$. Clearly $(0, j_1, 1) \in L$, since $(0,0,1) \notin L$, and 1 occurs exactly twice in row 0. Also $e_2 \neq 2$, as 2 also occurs exactly twice in row 0 of A_P, and $(0,0,2), (0, j_1, e_2) \notin L$.

If $e_2 = 0$ we are done, otherwise in general let j_d be the column such that $(0, j_d, e_d) \in L$ and let $\{e_{d+1}\} = (0, j_d)_{A_P} \setminus \{e_d\}$, for $d \geq 2$. We cannot have $e_{d+1} = 2$, since $(0,0,2)$, $(0, j_d, e_{d+1}) \notin L$, and 2 occurs exactly twice in row 0.

Each e_d is distinct, and we terminate at $e_z = 0$. □

Without loss of generality, relabel the columns of P (*and* corresponding columns of L), so that $j_d = d$ (or more specifically $(0,d)_{A_P} = \{e_d, e_{d+1}\}$, for $1 \leq d < z$). Also, if there are $z + y$ non-asterisked cells in row 0, rearrange the columns of P (and corresponding columns of L) so that these are $(0,0), (0,1), \ldots, (0, z+y-1)$.

Definition 8. Define partial binary operations $\star$ and $\bullet$ on $N \times N$ as follows:

$$i \star j = \begin{cases} (i,j)_L \text{ if } (i,j)_{A_P} \neq *, \\ \text{not defined otherwise.} \end{cases}$$

$$i \bullet j = \begin{cases} 2 \text{ if } (i,j) = (0,0), \\ k, \text{ where } \{k\} = (i,j)_{A_P} \setminus (i,j)_L \text{ if } (i,j)_{A_P} \neq * \text{ and } (i,j) \neq (0,0), \\ \text{not defined otherwise.} \end{cases}$$

Lemma 9. *Let operations $\star$ and $\bullet$ be as in the previous definition. Then:*

1. *$i \star j \neq i \bullet j$ for all i, j such that $(i,j)_{A_P} \neq *$,*
2. *$0 \star d = e_d$ for $0 \leq d < z$,*
3. *$0 \bullet d = e_{d+1}$ for $1 \leq d < z$,*
4. *there are unique rows r and s such that $r \neq s$, $r, s \neq 0$ and $r \bullet 0 = s \bullet 0 = 0$,*
5. *there is a unique column q such that $q \geq z$ and $0 \bullet q = 0$,*
6. *if $(i,j)_{A_P} \neq *$ and $(i,j) \notin \{(0,0), (\alpha, 0)\}$, where $\alpha \star 0 = 1$, there exists a unique row $i' \neq i$ such that $i' \bullet j = i \star j$,*
7. *if $(i,j)_{A_P} \neq *$ and $(i,j) \notin \{(0,0), (0,1)\}$, there exists a unique column $j' \neq j$ such that $i \bullet j' = i \star j$, and*
8. *if $(i,j)_{A_P} \neq *$, there exists a unique column $j' \neq j$ such that $i \star j' = i \bullet j$, and a unique row $i' \neq i$ such that $i' \star j = i \bullet j$.*

Proof. Conditions 1, 2 and 3 are easy to verify. ¿From Lemma 6, 0 occurs 3 times in column 0 in A_P, and 3 times in row 0 in A_P. Condition 4 follows from the former.

¿From Conditions 2 and 3, and the fact that $e_1, e_2, \ldots, e_z$ are all distinct, 0 occurs twice in cells from $(0,0)$ up to $(0, z-1)$ in A_P. In fact, $0 \in (0,0)_{A_P}$ and $0 \in (0, z-1)_{A_P}$. Therefore there must be a unique column $q \geq z$ such that $0 \bullet q = 0$.

Conditions 6, 7 and 8 follow from the fact that A_P is flexible and using simple counting arguments. □

Definition 10. Let T be the partial latin square contained in $L \times L$ with shape:

$$\{((0,j),(k,l)) \mid 0 \leq k \leq y+z-1, (j,l) \in (\mathcal{S}_{L \setminus P} \setminus \{(0,d) \mid 1 \leq d \leq z-1\})\} \cup$$
$$\{((i,0),(k,l)) \mid 0 \leq l \leq z-1, (i,k) \notin (\mathcal{S}_{L \setminus P} \setminus \{(0,d) \mid z \leq d \leq y+z-1\})\}.$$

Let $(i,j) \circ_T (k,l)$ denote the entry in cell $((i,j),(k,l))$ of T.

Lemma 11. *Let T and $\circ_T$ be defined as above. Then,*

1. $(i,j) \circ_T (k,l) = (i \star k, j \star l)$, *for each* $((i,j),(k,l)) \in S_T$, *and*
2. $S_T \subseteq S_{L \times L} \setminus S_{P \otimes P}$.

Proof. Condition 1 follows from the definition of $L \times L$. Condition 2 follows from the fact that cells $(0,0)$, $(0,1), \ldots (0, y+z-1)$ are empty in P. □

Note in Lemma 9, the operations $\star$ and $\bullet$ taken over the shape $S_L \setminus S_P$ *almost* correspond to a latin trade with a disjoint mate. In a very hand-wavey fashion we might think of this as a "near" latin trade.

In [15], Donovan and Mahmoodian show how any latin trade may be considered as a sum of intercalates, where addition is carefully defined. We employ the same principle in this paper, except we take the sum of "near" latin trades. In the processing of adding these "near" latin trades, the parts that *do not* adhere to the conditions of a latin trade will disappear.

We will eventually show that the partial latin square T given in Definition 10 is a latin trade. Essentially what we are doing is taking $z+y$ copies of the "near" latin trade for $i=0$, and z copies of the "near" latin trade for $j=0$ and combining them as in [15]. For a concrete example we encourage the reader to look ahead to Example 18 to see the process in action.

In the next definition, as in Condition 4 of Lemma 9, let $r, s \neq 0$, $r \neq s$ be the unique rows such that $(r \bullet 0) = (s \bullet 0) = 0$. Also, let r_d be the unique row such that $r_d \bullet d = 0 \star d$, where $1 \leq d \leq y+z-1$. (Such a row must exist because of Condition 6 of Lemma 9.)

Definition 12. Let $\circ_{T'}$ be an operation defined only on cells in the shape of T. We split the definition of T' into eight cases, which will be referred to in future lemmas.

Case 1: $i=j=0$, $0 \leq k \leq z-1$, $1 \leq l \leq z-1$. $(0,0) \circ_{T'} (k,l) = (e_{k+1}, 0 \star l)$.

Case 2: $i=0, j=r, l=0$, $0 \leq k \leq z-1$. $(0,r) \circ_{T'} (k,0) = (e_{k+1}, 0)$.

Case 3: $i=0, j=s, l=0$, $k=0$ or $z \leq k \leq y+z-1$. $(0,s) \circ_{T'} (k,0) = (0 \bullet k, 0)$.

Case 4: $i=0, 1 \leq l \leq z-1$, $j=r_l$, ($k=0$ or $z \leq k \leq y+z-1$). $(0,r_l) \circ_{T'} (k,l) = (0 \bullet k, 0 \star l)$.

Case 5: $i=0$, excluding Cases 1 through to 4. $(i,j) \circ_{T'} (k,l) = (i \star k, j \bullet l)$.

Case 6: $i=r>0$, $j=k=0$, $0 \leq l \leq z-1$. $(r,0) \circ_{T'} (0,l) = (0, e_{l+1})$.

Case 7: $1 \le k \le y+z-1$, $i = r_k > 0$, $j = 0$, $0 \le l \le z-1$. $(r_k, 0) \circ_{T'} (k, l) = (0 \star k, e_{l+1})$.

Case 8: $i > 0$, excluding Cases 6 and 7. $(i, j) \circ_{T'} (k, l) = (i \bullet k, j \star l)$.

This defines T'.

We wish to show, eventually, that T is a latin trade, with entries in the disjoint mate given by the operation $\circ_{T'}$. First we show that the operations $\circ_T$ and $\circ_{T'}$ never coincide.

Lemma 13. *$(i, j) \circ_T (k, l) \ne (i, j) \circ_{T'} (k, l)$ for each $((i, j), (k, l)) \in \mathcal{S}_T$.*

Proof. We run through each of the cases in the definition of T'. For Cases 1 and 2, we use the fact that $0 \star k = e_k$ (Condition 2 from Lemma 9) and $e_k \ne e_{k+1}$. For Cases 3 and 4, observe that $0 \star k \ne 0 \bullet k$ (Condition 1 of Lemma 9). For Case 5, $j \star l \ne j \bullet l$. For Cases 6 and 7, we need the fact that $0 \star l = e_l \ne e_{l+1}$. Finally for Case 8, $i \star k \ne i \bullet k$. □

Lemma 14. *For each cell $((i, j), (k, l)) \in S_T$, there exists a cell $((i, j), (k', l')) \in S_T$ such that $(i, j) \circ_{T'} (k', l') = (i, j) \circ_T (k, l)$.*

Proof. The proof involves numerous cases, but each case can be directly verified from previous lemmas. The 'R' in this proof stands for 'Row', since we are showing that the rows of T are balanced. *The terms "Case 1", "Case 2", and so on, will always refer to cases from Definition* 12.

R1: $i = 0$, $j = 0$. ¿From Definition 10 either $0 \le k \le y+z-1$, and ($l = 0$ or $z \le l \le y+z-1$) or $0 \le k \le z-1$ and $0 \le l \le z-1$.
- (a) $k = 0$ and $l = 0$. ¿From Condition 5 of Lemma 9, $0 \bullet q = 0$ for some $q \ge z$. Then $(0, 0) \circ_{T'} (0, q) = (0, 0)$, from Case 5. But $(0, 0) = (0, 0) \circ_T (0, 0)$.
- (b) $k = 0$ and $1 \le l \le z-1$. Then $(0, 0) \circ_{T'} (z-1, l) = (e_z, 0 \star l)$, from Case 1. But $(e_z, 0 \star l) = (0, 0 \star l) = (0, 0) \circ_T (0, l)$.
- (c) $0 \le k \le y+z-1$ and $z \le l \le y+z-1$. Here $0 \star l \notin \{e_1, e_2, \dots e_z\}$ so from Condition 7 of Lemma 9 there exists a unique column f such that either and $0 \star l = 0 \bullet f$. Moreover, either $f = 0$ or $f \ge z$. Then $(0, 0) \circ_{T'} (k, f) = (0 \star k, 0 \star l)$, from Case 5.
- (d) $0 < k \le z-1$ and $l = 0$. Here $(0, 0) \circ_{T'} (k, q) = (0 \star k, 0 \bullet q)$ by Case 5. (Remember $q \ge z$ from Condition 5 of Lemma 9.) Now $(0 \star k, 0 \bullet q) = (0 \star k, 0) = (0, 0) \circ_T (k, 0)$.
- (e) $z \le k \le y+z-1$ and $l = 0$. Here $(0, 0) \circ_{T'} (k, z-1) = (0 \star k, e_z)$ by Case 5. Now $(0 \star k, e_z) = (0 \star k, 0) = (0, 0) \circ_T (k, 0)$.
- (f) $0 < k \le z-1$ and $0 < l \le z-1$. Here $(0, 0) \circ_{T'} (k-1, l) = (e_k, 0 \star l)$ (from Case 1), and $(e_k, 0 \star l) = (0, 0) \circ_T (k, l)$.

R2: $i = 0$, $j = r$. Then $0 \le k \le y+z-1$ and $(r, l) \notin S_P$. Let β be the unique column such that $r \star \beta = 0$. (This exists because $r \bullet 0 = 0$ and Condition 8 of Lemma 9.)
- (a) $k = 0$ and $l = \beta$. Then $(0, r) \circ_{T'} (z-1, 0) = (0, 0)$ by Case 2, and $(0, 0) = (0, r) \circ_T (0, \beta)$.
- (b) $0 < k \le z-1$ and $l = \beta$. Then $(0, r) \circ_{T'} (k-1, 0) = (e_k, 0)$ by Case 2, and $(e_k, 0) = (0, r) \circ_T (k, \beta)$.
- (c) $z \le k \le y+z-1$ and $l = \beta$. Then $(0, r) \circ_{T'} (k, 0) = (0 \star k, 0)$ by Case 5, and $(0 \star k, 0) = (0, r) \circ_T (k, \beta)$.
- (d) $l \ne \beta$. Then $r \star l = r \bullet f$ for a unique column f, by Condition 7 of Lemma 9. Moreover $f \ne 0$, since $l \ne \beta$. So from Case 5, $(0, r) \circ_{T'} (k, f) = (0 \star k, r \bullet f) = (0 \star k, r \star l) = (0, r) \circ_T (k, l)$.

R3: $i = 0$, $j = s$. Then $0 \le k \le y+z-1$ and $(s, l) \notin S_P$. Let β' be the unique column such that $s \star \beta' = 0$. (This exists because $s \bullet 0 = 0$, and Condition 8 of Lemma 9.)

(a) $k = 0$, $l = \beta'$, $z \le k \le y + z - 1$, $l = \beta'$. ¿From Condition 5 of Lemma 9, we have $0 \bullet q = 0$ for some $q \ge z$. So, $(0, s) \circ_{T'} (q, 0) = (0 \bullet q, 0)$ (Case 3), and $(0 \bullet q, 0) = (0, 0) = (0, s) \circ_T (0, \beta')$.

(b) $0 < k \le z - 1$, $l = \beta'$. Then $(0, s) \circ_{T'} (k, 0) = (0 \star k, 0)$ by Case 5, and $(0 \star k, 0) = (0, s) \circ_T (k, \beta')$.

(c) $z \le k \le y + z - 1$, $l = \beta'$. As in Case R1c, there exists a unique column f such that $0 \star k = 0 \bullet f$, where $f = 0$ or $f \ge z$. Then, $(0, s) \circ_{T'} (f, 0) = (0 \bullet f, 0)$ (Case 3), and $(0 \bullet f, 0) = (0 \star k, 0) = (0, s) \circ_T (k, \beta')$.

(d) $l \ne \beta'$. Then $s \star l = s \bullet f$ for a unique column f by Condition 7 of Lemma 9. Moreover $f \ne 0$, since otherwise $s \star l = 0$ and $l = \beta'$, a contradiction. So from Case 5, $(0, s) \circ_{T'} (k, f) = (0 \star k, s \bullet f) = (0 \star k, s \star l) = (0, s) \circ_T (k, l)$.

R4: $i = 0$, $1 \le d \le z - 1$, $j = r_d$. Then $0 \le k \le y + z - 1$ and $(r_d, l) \notin \mathcal{S}_P$. Let β_d be the unique column such that $r_d \star \beta_d = e_d (= 0 \star d)$. (This exists because $r_d \bullet d = e_d$, and Condition 8 of Lemma 9.)

(a) $k = 0$, $l = \beta_d$. ¿From Condition 5 of Lemma 9, we have $0 \bullet q = 0$ for some $q \ge z$. Then $(0, r_d) \circ_{T'} (q, d) = (0 \bullet q, 0 \star d)$ by Case 4. But $(0 \bullet q, 0 \star d) = (0, e_d) = (0, r_d) \circ_T (0, \beta_d)$.

(b) $0 < k \le z - 1$, $l = \beta_d$. Then $(0, r_d) \circ_{T'} (k, d) = (0 \star k, r_d \bullet d)$ by Case 5. And $(0 \star k, r_d \bullet d) = (0 \star k, e_d) = (0, r_d) \circ_T (k, \beta_d)$.

(c) $z \le k \le y + z - 1$, $l = \beta_d$. As in Case R1c, there exists a unique column f such that $0 \star k = 0 \bullet f$, where $f = 0$ or $f \ge z$. Then, $(0, r_d) \circ_{T'} (f, d) = (0 \bullet f, 0 \star d)$ (Case 4), and $(0 \bullet f, 0 \star d) = (0 \star k, e_d) = (0, r_d) \circ_T (k, \beta_d)$.

(d) $l \ne \beta_d$. Then $r_d \star l = r_d \bullet f$ for a unique column f by Condition 7 of Lemma 9. Moreover $f \ne d$, since otherwise $r_d \star l = e_d$ and $l = \beta_d$, a contradiction. So from Case 5, $(0, r_d) \circ_{T'} (k, f) = (0 \star k, r_d \bullet f) = (0 \star k, r_d \star l) = (0, r_d) \circ_T (k, l)$.

R5: $i = 0$, $j \notin \{0, r, s, r_d \mid 1 \le d \le z - 1\}$. Then, from the shape of T, we have $0 \le k \le y + z - 1$ and $(j, l) \notin \mathcal{S}_P$. Let f be the unique column such that $j \bullet f = j \star l$, by Condition 7 of Lemma 9. Then $(0, j) \circ_{T'} (k, f) = (0 \star k, j \bullet f)$ by Case 5, and $(0 \star k, j \bullet f) = (0 \star k, j \star l) = (0, j) \circ_T (k, l)$.

Whew! For the remaining cases $i > 0$. ¿From the shape of T, this means that $j = 0$, $0 \le l \le z - 1$ and $(i, k) \notin \mathcal{S}_P$.

R6: $i = r$. As in Case R2, let β be the unique column such that $r \star \beta = 0$.

(a) $k = \beta$, $l = 0$. Then $(r, 0) \circ_{T'} (0, z-1) = (0, 0)$ by Case 6, and $(0, 0) = (r, 0) \circ_T (\beta, 0)$.

(b) $k = \beta$, $0 < l \le z - 1$. Then $(r, 0) \circ_{T'} (0, l - 1) = (0, e_l)$ by Case 6, and $(0, e_l) = (r, 0) \circ_T (\beta, l)$.

(c) $k \ne \beta$. By Condition 7 of Lemma 9 there is a unique column f such that $r \bullet f = r \star k$. Since $k \ne \beta$, then $r \star k \ne 0$, and thus $f \ne 0$. Then $(r, 0) \circ_{T'} (f, l) = (r \bullet f, 0 \star l)$ by Case 8, and $(r \bullet f, 0 \star l) = (r \star k, 0 \star l) = (r, 0) \circ_T (k, l)$.

R7: $1 \le d \le y + z - 1$, $i = r_d$. As in Case R4, let β_d be the unique column such that $r_d \star \beta_d = 0 \star d$.

(a) $k = \beta_d$, $l = 0$. Then $(r_d, 0) \circ_{T'} (d, z - 1) = (0 \star d, 0)$ by Case 7, and $(0 \star d, 0) = (r_d, 0) \circ_T (\beta_d, 0)$.

(b) $k = \beta_d$, $0 < l \le z - 1$. Then $(r_d, 0) \circ_{T'} (d, l - 1) = (0 \star d, e_l)$ by Case 7, and $(0 \star d, e_l) = (r_d, 0) \circ_T (\beta_d, l)$.

(c) $k \ne \beta_d$. Since $k \ne \beta_d$, we have $r_d \star k \ne e_d$, and there is a column $f \ne d$ such that $r_d \bullet f = r_d \star k$ (by Condition 7 of Lemma 9). Then $(r_d, 0) \circ_{T'} (f, l) = (r_d \bullet f, 0 \star l)$ by Case 8, and $(r_d \bullet f, 0 \star l) = (r_d \star k, 0 \star l) = (r_d, 0) \circ_T (k, l)$.

R8: $i \notin \{0, r, r_d \mid 1 \le d \le y + z - 1\}$. Then, from the shape of T, we have $j = 0$, $0 \le l \le z - 1$ and $(i, k) \notin \mathcal{S}_P$. Let f be the unique column such that $i \bullet f = i \star k$

(by Condition 7 of Lemma 9). Then $(i,0) \circ_{T'} (f,l) = (i \bullet f, 0 \star l)$ by Case 8, and $(i \bullet f, 0 \star l) = (i \star k, 0 \star l) = (i,0) \circ_T (k,l)$.

□

In the next lemma we also make use of facts from Lemma 9.

Lemma 15. *For each cell* $((i,j),(k,l)) \in \mathcal{S}_T$*, there exists a cell* $((i',j'),(k,l)) \in \mathcal{S}_T$ *such that* $(i',j') \circ_{T'} (k,l) = (i,j) \circ_T (k,l)$.

Proof. The 'C' in this lemma stands for 'Column', since we are showing that the columns of T are balanced. *The terms "Case 1", "Case 2", and so on, will always refer to cases from Definition* 12.

C1: $k = 0$, $l = 0$. Then from the shape of T, either $i = 0$ and $(j,0) \notin \mathcal{S}_P$, or $j = 0$ and $(i,0) \notin \mathcal{S}_P$.
- (a) $i = 0$ and $j = 0$. Then $(s,0) \circ_{T'} (0,0) = (s \bullet 0, 0 \star 0)$ by Case 8, and $(s \bullet 0, 0 \star 0) = (0,0) = (0,0) \circ_T (0,0)$.
- (b) $i = 0$, $j \star 0 = 2$. Such a j must exist because $0 \bullet 0 = 2$. Then, $(0,0) \circ_{T'} (0,0) = (0 \star 0, 0 \bullet 0)$ (by Case 5), and $(0 \star 0, 0 \bullet 0) = (0,2) = (0,j) \circ_T (0,0)$.
- (c) $i = 0$, $j \star 0 = 1$. Then, $(r,0) \circ_{T'} (0,0) = (0, e_1)$ (by Case 6), and $(0,e_1) = (0 \star 0, 1) = (0,j) \circ_T (0,0)$.
- (d) $i = 0$, $j \star 0 \notin \{0,1,2\}$. Let f be the unique row such that $f \bullet 0 = j \star 0$. Then $f \neq r$, $f \neq s$ and $f \neq 0$. So by Case 5, $(0,f) \circ_{T'} (0,0) = (0 \star 0, f \bullet 0) = (0, j \star 0) = (0,j) \circ_T (0,0)$.
- (e) $j = 0$, $i \star 0 = 2$. Then by Case 3, $(0,s) \circ_{T'} (0,0) = (0 \bullet 0, 0) = (2,0) = (i,0) \circ_T (0,0)$.
- (f) $j = 0$, $i \star 0 = 1$. Then by Case 2, $(0,r) \circ_{T'} (0,0) = (e_1,0) = (1,0) = (i,0) \circ_T (0,0)$.
- (g) $j = 0$, $i \star 0 \notin \{0,1,2\}$. Let f be the unique row such that $f \bullet 0 = i \star 0$. Note that $f \notin \{0,r,s\}$. Then by Case 8, $(f,0) \circ_{T'} (0,0) = (f \bullet 0, 0) = (i \star 0, 0) = (i,0) \circ_T (0,0)$.

C2: $k = 0$, $l > 0$. Then from the shape of T, either $i = 0$ and $(j,l) \notin \mathcal{S}_P$, or $j = 0$, $0 \le l \le z-1$ and $(i,0) \notin \mathcal{S}_P$.
- (a) $1 \le l \le z-1$, $i = j = 0$. Then $(s,0) \circ_{T'} (0,l) = (s \bullet 0, 0 \star l)$ by Case 8, and $(s \bullet 0, 0 \star l) = (0,l) = (0,0) \circ_T (0,l)$.
- (b) $1 \le l \le z-1$, $i = 0$, $j \star l = e_{l+1}$. Note that such a j exists because $0 \bullet l = e_{l+1}$. Then, $(r,0) \circ_{T'} (0,l) = (0, e_{l+1})$ by Case 6, and $(0, e_{l+1}) = (0,j) \circ_T (0,l)$.
- (c) $1 \le l \le z-1$, $i = 0$, $j \star l \neq e_{l+1}$, $j \neq 0$. Then $j \star l = f \bullet l$ for some row $f \notin \{0, r_l\}$. By Case 5, $(0,f) \circ_{T'} (0,l) = (0 \star 0, f \bullet l) = (0, j \star l) = (0,j) \circ_T (0,l)$.
- (d) $1 \le l \le z-1$, $j = 0$, $i \star 0 = 1$. Then from Case 1, $(0,0) \circ_{T'} (0,l) = (e_1, 0 \star l) = (1, 0 \star l) = (i,0) \circ_T (0,l)$.
- (e) $1 \le l \le z-1$, $j = 0$, $i \star 0 = 2$. Then from Case 4, $(0,r_l) \circ_{T'} (0,l) = (0 \bullet 0, 0 \star l) = (2, 0 \star l) = (i,0) \circ_T (0,l)$.
- (f) $1 \le l \le z-1$, $j = 0$, $i \star 0 \notin \{0,1,2\}$. Then there is a unique row f such that $f \bullet 0 = i \star 0$. Note that $f \notin \{0,r,s\}$. Then from Case 8, $(f,0) \circ_{T'} (0,l) = (f \bullet 0, 0 \star l) = (i \star 0, 0 \star l) = (i,0) \circ_T (0,l)$.
- (g) $z \le l \le y+z-1$. Then we must have $i = 0$ and $(j,l) \notin \mathcal{S}_P$. Let f be the unique row such that $f \bullet l = j \star l$. Then by Case 5, $(0,f) \circ_{T'} (0,l) = (0 \star 0, f \bullet l) = (0, j \star l) = (0,j) \circ_T (0,l)$.

C3: $l = 0$, $k > 0$. Then from the shape of T, either $i = 0$ and $(j,0) \notin \mathcal{S}_P$, or $j = 0$ and $(i,k) \notin \mathcal{S}_P$.
- (a) $i = 0$, $0 < k \le z-1$, $j = 0$. Then by Case 5, $(0,s) \circ_{T'} (k,0) = (0 \star k, s \bullet 0) = (0 \star k, 0) = (0,0) \circ_T (k,0)$.
- (b) $i = 0$, $0 < k \le z-1$, $j \star 0 = 1$. Then by Case 7, $(r_k, 0) \circ_{T'} (k,0) = (0 \star k, e_1) = (0 \star k, 1) = (0,j) \circ_T (k,0)$.

(c) $i = 0$, $0 < k \le z-1$, $j \star 0 = 2$. Then by Case 8, $(r_k, j) \circ_{T'} (k,0) = (r_k \bullet k, j \star 0) = (0 \star k, j \star 0) = (0,j) \circ_T (k,0)$.

(d) $i = 0$, $0 < k \le z-1$, $j \star 0 \notin \{0,1,2\}$. Let f be the unique row such that $f \bullet 0 = j \star 0$. Now, $r \neq f$, as if $r = f$, then $j \star 0 = 0$ and $j = 0$. Similarly, $s \neq f$. Also $f \neq 0$ as $j \star 0 \neq 2$. Then from Case 5, $(0,f) \circ_{T'} (k,0) = (0 \star k, f \bullet 0) = (0 \star k, j \star 0) = (0,j) \circ_T (k,0)$.

(e) $i = 0$, $z \le k \le y+z-1$, $j = 0$. Then by Case 5, $(0,r) \circ_{T'} (k,0) = (0 \star k, r \bullet 0) = (0 \star k, 0) = (0,0) \circ_T (k,0)$.

(f) $i = 0$, $z \le k \le y+z-1$, $j \star 0 = 1$. Then by Case 7, $(r_k, 0) \circ_{T'} (k,0) = (0 \star k, e_1) = (0 \star k, 1) = (0,j) \circ_T (k,0)$.

(g) $i = 0$, $z \le k \le y+z-1$, $j \star 0 \neq 1$, $j \neq 0$. Let f be the unique row such that $f \bullet 0 = j \star 0$. Then since $j \neq 0$, $f \neq s$. So by Case 5, $(0,f) \circ_{T'} (k,0) = (0 \star k, f \bullet 0) = (0 \star k, j \star 0) = (0,j) \circ_T (k,0)$.

(h) $i > 0$, $j = 0$, $0 < k \le z-1$ such that $i \star k = 0 \bullet k$. Then, from Case 2, $(0,r) \circ_{T'} (k,0) = (e_{k+1}, 0) = (i,0) \circ_T (k,0)$.

(i) $i > 0$, $j = 0$, $z \le k \le y+z-1$ such that $i \star k = 0 \bullet k$. Then from Case 3, $(0,s) \circ_{T'} (k,0) = (0 \bullet k, 0) = (i,0) \circ_T (k,0)$.

(j) $i > 0$, $j = 0$, $0 < k \le y+z-1$ such that $i \star k \neq 0 \bullet k$. Let f be the unique row such that $f \bullet k = i \star k$. Since $i \star k \neq 0 \bullet k$, $f \neq 0$. Also, since $i \neq 0$, and since $0 \star k = r_k \bullet k$, $f \neq r_k$. Then by Case 8, $(f,0) \circ_{T'} (k,0) = (f \bullet k, 0) = (i \star k, 0) = (i,0) \circ_T (k,0)$.

C4: $l > 0$, $k > 0$. Then from the shape of T, either $i = 0$ and $(j,l) \notin \mathcal{S}_P$, or $j = 0$, $0 \le l \le z-1$ and $(i,k) \notin \mathcal{S}_P$.

(a) $i = 0$, $0 < l \le z-1$ such that $j \star l = 0 \bullet l$. Then by Case 7, $(r_k, 0) \circ_{T'} (k,l) = (0 \star k, 0 \bullet l) = (0,j) \circ_T (k,l)$.

(b) $i = 0$, $z \le l \le y+z-1$ such that $j \star l = 0 \bullet l$. Then by Case 5, $(0,0) \circ_{T'} (k,l) = (0 \star k, 0 \bullet l) = (0 \star k, j \star l) = (0,j) \circ_T (k,l)$.

(c) $i = 0$, $j \star l \neq 0 \bullet l$. Let f be the unique row such that $f \bullet l = j \star l$. Since $j \star l \neq 0 \bullet l$, $f \neq 0$. Also, since $j \neq 0$, $f \neq r_l$. Then by Case 5, $(0,f) \circ_{T'} (k,l) = (0 \star k, f \bullet l) = (0 \star k, j \star l) = (0,j) \circ_T (k,l)$.

(d) $j = 0$, $i > 0$, $0 < l \le z-1$, $0 < k \le z-1$ such that $i \star k = 0 \bullet k$. Recall that $0 \bullet k = e_{k+1}$ for $0 < k \le z-1$. ¿From Case 1, $(0,0) \circ_{T'} (k,l) = (e_{k+1}, 0 \star l) = (i \star k, 0 \star l) = (i,0) \circ_T (k,l)$.

(e) $j = 0$, $i > 0$, $0 < l \le z-1$, $z \le k \le y+z-1$ such that $i \star k = 0 \bullet k$. ¿From Case 4, $(0, r_k) \circ_{T'} (k,l) = (0 \bullet k, 0 \star l) = (i \star k, 0 \star l) = (i,0) \circ_T (k,l)$.

(f) $j = 0$, $i > 0$, $0 < l \le z-1$, $i \star k \neq 0 \bullet k$. Now, let f be the unique row such that $f \bullet k = i \star k$. If $f = r_k$, then $i = 0$, a contradiction. So by Case 8, $(f,0) \circ_{T'} (k,l) = (f \bullet k, 0 \star l) = (i \star k, 0 \star l) = (i,0) \circ_T (k,l)$.

This completes every case!! □

Lemma 16. *There exists a partial latin square, denoted by T', with the same shape and order as T and cell $((i,j),(k,l))$ containing entry $(i,j) \circ_{T'} (k,l)$.*

Proof. We need to show that T' is a well-defined partial latin square (i.e. there are no repeated entries in rows or columns). This follows from Lemmas 14, 15 and the fact that T is a well-defined partial latin square. □

Theorem 17. *Let P be a uniquely completable partial latin square with an array of alternatives A_P such that exactly one cell of A_P has a set of size three, and remaining cells contain either a set of size two or an asterisk. Then the completable product $P \otimes P$ is not uniquely completable to $L \times L$.*

Proof. Recall from Lemma 11 that $S_T \subseteq S_{L\times L} \setminus S_{P\otimes P}$. So all we need to show is that T is a latin trade and T' its disjoint mate. ¿From the previous lemma T' is a partial latin square with the same shape as T. ¿From Lemma 14, the rows of T and T' are mutually balanced. ¿From Lemma 15, the columns of T and T' are mutually balanced. Finally from Lemma 13, T and T' are disjoint. □

Example 18. Consider the following totally weak, UC partial latin square P, together with its unique completion to a latin square L.

$P =$

			4	3
	2			
		1		4
			0	2
3	4			

and $L =$

0	1	2	4	3
4	2	3	1	0
2	0	1	3	4
1	3	4	0	2
3	4	0	2	1

Consider the array of alternatives A_P.

$A_P =$

$\{0,1,2\}$	$\{0,1\}$	$\{0,2\}$	$*$	$*$
$\{0,1,4\}$	$*$	$\{0,3,4\}$	$\{1,3\}$	$\{0,1\}$
$\{0,2\}$	$\{0,3\}$	$*$	$\{2,3\}$	$*$
$\{1,4\}$	$\{3,1\}$	$\{4,3\}$	$*$	$*$
$*$	$*$	$\{0,2\}$	$\{1,2\}$	$\{0,1\}$

Using the Bedford-Whitehouse methods of "semi-forcing" elements we can simplify the array of alternatives to the following.

$A'_P =$

	0	1	2	3	4
0	$\{0,1,2\}$	$\{0,1\}$	$\{0,2\}$	$*$	$*$
1	$\{0,4\}$	$*$	$\{3,4\}$	$\{1,3\}$	$\{0,1\}$
2	$\{0,2\}$	$\{0,3\}$	$*$	$\{2,3\}$	$*$
3	$\{1,4\}$	$\{3,1\}$	$\{4,3\}$	$*$	$*$
4	$*$	$*$	$\{0,2\}$	$\{1,2\}$	$\{0,1\}$

Since exactly one cell has a set of size 3 and remaining cells have either a set of size 2 or an asterisk $*$, we can apply Theorem 17.

First we give tables for the operations $\star$ and $\bullet$, as given in Definition 8.

$\star$	0	1	2	3	4
0	0	1	2		
1	4		3	1	0
2	2	0		3	
3	1	3	4		
4			0	2	1

$\bullet$	0	1	2	3	4
0	2	0	0		
1	0		4	3	1
2	0	3		2	
3	4	1	3		
4			2	1	0

Also note that $z = 2$, $y = 1$, $r = 1$, $s = 2$, $r_1 = 3$ and $r_2 = 4$. The latin trade T from $(L \times L) \setminus (P \otimes P)$ is given below, together with its disjoint mate T', thus proving $P \otimes P$ is not uniquely completable.

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,0)	(3,1)	(4,0)	(4,1)
(0,0)	(0,0)	(0,1)	(0,2)			(1,0)	(1,1)	(1,2)			(2,0)		(2,2)						
(0,1)	(0,4)		(0,3)	(0,1)	(0,0)	(1,4)		(1,3)	(1,1)	(1,0)	(2,4)		(2,3)	(2,1)	(2,0)				
(0,2)	(0,2)	(0,0)		(0,3)		(1,2)	(1,0)		(1,3)		(2,2)	(2,0)		(2,3)					
(0,3)	(0,1)	(0,3)	(0,4)			(1,1)	(1,3)	(1,4)			(2,1)	(2,3)	(2,4)						
(0,4)			(0,0)	(0,2)	(0,1)			(1,0)	(1,2)	(1,1)			(2,0)	(2,2)	(2,1)				
(1,0)	(4,0)	(4,1)									(3,0)	(3,1)				(1,0)	(1,1)	(0,0)	(0,1)
(2,0)	(2,0)	(2,1)				(0,0)	(0,1)									(3,0)	(3,1)		
(3,0)	(1,0)	(1,1)				(3,0)	(3,1)				(4,0)	(4,1)							
(4,0)											(0,0)	(0,1)				(2,0)	(2,1)	(1,0)	(1,1)

Figure 2: The latin trade T

	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,0)	(3,1)	(4,0)	(4,1)
(0,0)	(0,2)	(1,1)	(0,0)			(1,2)	(0,1)	(1,0)			(2,2)		(2,0)						
(0,1)	(1,0)		(0,4)	(0,3)	(0,1)	(0,0)		(1,4)	(1,3)	(1,1)	(2,0)		(2,4)	(2,3)	(2,1)				
(0,2)	(2,0)	(0,3)		(0,2)		(1,0)	(1,3)		(1,2)		(0,0)	(2,3)		(2,2)					
(0,3)	(0,4)	(2,1)	(0,3)			(1,4)	(1,1)	(1,3)			(2,4)	(0,1)	(2,3)						
(0,4)			(0,2)	(0,1)	(0,0)			(1,2)	(1,1)	(1,0)			(2,2)	(2,1)	(2,0)				
(1,0)	(0,1)	(0,0)									(4,0)	(4,1)				(3,0)	(3,1)	(1,0)	(1,1)
(2,0)	(0,0)	(0,1)				(3,0)	(3,1)									(2,0)	(2,1)		
(3,0)	(4,0)	(4,1)				(1,1)	(1,0)				(3,0)	(3,1)							
(4,0)											(2,1)	(2,0)				(1,0)	(1,1)	(0,0)	(0,1)

Figure 3: The disjoint mate T'

Corollary 19. *Let P be a totally weak, uniquely completable partial latin square (to a latin square L), with an array of alternatives A_P. Let $\mathcal{B}$ be an array square with $\mathcal{S}_{\mathcal{B}} \subseteq \mathcal{S}_{A_P}$, and*

$(i,j)_L \in (i,j)_{\mathcal{B}} \subseteq (i,j)_{A_P}$ *for each* (i,j) *such that* $(i,j)_{\mathcal{B}} \neq *$. *Let* $\mathcal{B}$ *have the following properties.*

1. *Exactly one cell has a set of size* 3.
2. *Every remaining cell contains either a set of size* 2 *or an asterisk.*

Then the partial latin square $P \times P$ *is not uniquely completable.*

Example 20. Consider the following totally weak, UC partial latin square P, together with its unique completion to a latin square L.

$P =$

				4
	0			
2		4		
3		1		
			1	3

and $L =$

0	1	2	3	4
1	0	3	4	2
2	3	4	0	1
3	4	1	2	0
4	2	0	1	3

Consider the array of alternatives $\mathcal{A}$.

$A_P =$

$\{0,1\}$	$\{1,2,3\}$	$\{0,2,3\}$	$\{0,2,3\}$	$*$
$\{1,4\}$	$*$	$\{2,3\}$	$\{2,3,4\}$	$\{1,2\}$
$*$	$\{1,3\}$	$*$	$\{0,3\}$	$\{0,1\}$
$*$	$\{2,4\}$	$*$	$\{0,2,4\}$	$\{0,2\}$
$\{0,4\}$	$\{2,4\}$	$\{0,2\}$	$*$	$*$

Using the Bedford-Whitehouse methods of "semi-forcing" elements we can simplify the array of alternatives to the following.

$A'_P =$

$\{0,1\}$	$\{1,3\}$	$\{0,2,3\}$	$\{2,3\}$	$*$
$\{1,4\}$	$*$	$\{2,3\}$	$\{2,3,4\}$	$\{1,2\}$
$*$	$\{1,3\}$	$*$	$\{0,3\}$	$\{0,1\}$
$*$	$\{2,4\}$	$*$	$\{0,2,4\}$	$\{0,2\}$
$\{0,4\}$	$\{2,4\}$	$\{0,2\}$	$*$	$*$

Then we observe that this reduced array of alternatives contains the following array square $\mathcal{B}$:

$\mathcal{B} =$

$\{0,1\}$	$\{1,3\}$	$\{0,2\}$	$\{2,3\}$	$*$
$\{1,4\}$	$*$	$*$	$\{2,4\}$	$\{1,2\}$
$*$	$\{1,3\}$	$*$	$\{0,3\}$	$\{0,1\}$
$*$	$\{2,4\}$	$*$	$\{0,2,4\}$	$\{0,2\}$
$\{0,4\}$	$\{2,4\}$	$\{0,2\}$	$*$	$*$

¿From Corollary 19, $P \times P$ is not UC.

In the following lemma, a critical set is defined to be a uniquely completable set that is minimal with respect to this property.

Lemma 21. *If* P *is a Bedford and Whitehouse totally weak critical set of order at most* 5, *then the completable product* $P \otimes P$ *has more than one completion.*

Proof. According to [1], there are two Bedford and Whitehouse totally weak critical sets of order at most 5 (up to isotopism). One of these is P from Example 18; the other is P from Example 20. □

We conclude the paper by posing the following open problem.

Does there exist a Bedford and Whitehouse totally weak uniquely completable partial latin square P, with a completable product $P \otimes P$ that is also uniquely completable?

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