Friendship 3-Hypergraphs

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1 Introduction

A *friendship* graph is a graph in which any two vertices have exactly one common neighbour. Introduced by Erdös, Rényi, and Sós.

An example is the windmill graph - a central vertex joined to every other vertex which are joined in pairs. The center vertex is called the politician whose friends with everyone. The graph is called the *universal friend graph*. It only exists for an odd number of vertices.

Theorem: In a friendship graph, there is a vertex that is joined to all other vertices.

There are beautiful proofs by Aigner and Ziegler, West, Longyear and Parsons, Huneke and an Olympiad website in New Zealand.

The proof of the theorem given here closely follows the write-up for the Olympiad preperation in New Zealand. This is based on earlier work by Huneke (2202) and much earlier work by Longyear and Parsons (1972).

The next lemma is all important in the proof of the theorem.

Lemma: In a friendship graph there is no simple cycle of length 4.
Proof: By definition of friendship graph.

Proof:

1) Assume graph, G is a not a universal friend graph.

2) Prove G is regular.

3) Either every pair of vertices are joined and we are done or

4) there exist non-joined vertices u and v where k=d(u) $\geq$ d(v).

5) Let u be adjacent to $w_1, w_2, \ldots, w_k$.

6) Let v and u have common neighbour say w_2.

7) Let u and w_2 have common neighbour, say w_1.

8) Also no w_i is joined to two other w_i's.

9) Then v is not joined to w_i for $i > 2$ to avoid 4-cycles.

10) Let v and w_i have common neighbour z_i for
$i = 2, 3, \ldots, k$.

11) Note z's are not w's to avoid 4-cycles.
Note z's are distinct to avoid 4-cycles.

12) $\mathrm{d}(v) \geq \mathrm{d}(u)$.

13)$\mathrm{d}(v) = \mathrm{d}(u)$.

14) Any vertex, not w_2, is not adjacent to v or u - so has degree k also

15) w_2 is non-adjacent to some vertex - so has degree k also.

16) Now G is regular of degree k. Consider the vertices adjacent to a particular vertex r. Now consider the k^2 vertices that are adjacent to those k vertices. These k^2 vertices include every vertex once except for vertex r which is included k times. So G has $k^2 - k + 1 = N$ vertices.

17) k is even. (diagram)

18) Let p be a prime that divides $k - 1$. Let us count all circuits of length p with a distinguished first vertex (intermediate vertices do not have to be distinct). Call this number S. Each circuit can have p rotations producing distinct circuits with distinct first vertices. So $S = 0$ mod p.

19) Count sequences of nodes of length $p-1$: $v_0, v_1, \ldots, v_{p-2}$ where v_i is adjacent to v_{i-1} and v_{i+1} and v_i are not necessarily distinct. There are Nk^{p-2} of them.

20) Let K_1 count those sequences with $v_0 = v_{p-2}$ and K_2 count those with $v_0 \neq v_{p-2}$. So $K_1 + K_2 = Nk^{p-2}$.

21)If $v_0 = v_{p-2}$, then we can choose v_{p-1} in k different ways to transform the sequence into a circuit of length p with distinguished first vertex.

22) If $v_0 \neq v_{p-2}$, then there is only 1 vertex (friendship property) that can be used to transform the sequence into a circuit of length p with distinguished first vertex.

$$
\begin{aligned}
S &= kK_1 + K_2 \\
&= (k-1)K_1 + (K_1 + K_2) \\
&= (k-1)K_1 + Nk^{p-2} \\
&\equiv Nk^{p-2} \bmod p
\end{aligned}
$$

Since $N = k^2 - k + 1 = k(k-1) + 1 \equiv 1 \bmod p$ and since $k^{p-2} \equiv 1^{p-2} \equiv 1 \bmod p$

$S \equiv 1 \mod p$. Contradiction.

So the problem is dead —wait —

Generalizations
Another way to phrase the Friendship problem is to say the Friendship Graph has the property that any two vertices are joined by a unique path of length 2. So change the length. Change 2 to 1. Oh –Just a regular graph.

Let's make the length of the path bigger. Kotzig conjectured that there were no such graphs. In a series of papers, Kotzig, Kostochka, Xing and Hu, Yang et al. proved the conjecture true. We need a new generalization.

Another way to generalize the problem is to change "one common neighbour" to "mmon neighbours". Mertzios and Unger, using graph theoretic means give a fairly straightforward proof that these graphs are regular and hence are strongly regular srg(n, k, l, μ) with $\lambda = \mu = m$ which correspond to a symmetric balanced incomplete block designs.

Some non-trivial examples are the line graph of K_6 with $n = 15, k = 8m = 4$, the $K_4 \times K_4$ Shrikhande graph with $n = 16, k = 6m = 2$ and the halved 5-cube with $n = 16, k = 10, m = 6$ Clebsch graph. But then maybe strongly regular graphs are not your cup of tea.

Desperately, we consider infinite graphs. Let $G_0 = C_5$. Let G_{n+1} be obtained from G_n by adding a separate common neighbour to each pair of vertices that does not yet have one then $G = \cup_{n=0}^{\infty} G_i$ is a counterexample to the friendship theorem. Let's forget infinite graphs.

The only place left to generalize is to go to hypergraphs. Very recently (i.e. not in print yet (i.e. not written yet)) Yair Caro has generalized the problem to:

An r-hypergraph H(V,E) on n vertices belongs to the friendship family F(n,r,p,m,k) (where $r - 1 \geq p \geq 1, m \geq r - p, k \geq p$) if the following hold:

For every m-set M of vertices of V, there exists a unique disjoint k-set K of vertices of V such that for every $(r-p)$-subset of A of M and every p-subset B of K, $A \cup B$ is an edge in $E(H)$.

$r = 2$

$F(n, 2, 1, 2, 1)$ is the class of friendship graphs.

$F(n, 2, 1, m, k), m \geq 3, k \geq 1$ only consists of the graph K_{m+k}. See Carstens and Kruse or Sudolsky.

$F(n, 2, 1, 1, k)$ is the class of regular graphs.

$F(n, 2, 1, 2, k), k \geq 2$ is the class of strongly regular graphs. See Bose and Shrikhande.

For $r \geq 3$, there is not much known except for:

$F(n, 3, 1.3, 1)$ which has 3 papers written about them. More later.—

$F(n, r, p, m, k)$ always contains the trivial K^r_{m+k}

$F(n, r, p, r-p, p)$ r-hypergraphs from Stiener Systems $S(r-p, r, n)$.
The case $p = r-1, m = 1, k = r-1$ always gives a matching.

He has some constructions - some generalized from Sös and some not.

From Sös:

A Friendship 3-Hypergraph is a 3-hypergraph in which any 3 vertices (elements), u, v and w, occur in pairs with a unique fourth element x; i.e., uvx, uwx, vwx are 3-hyperedges. The element x is said to *complete* the elements u, v and w.

How many of the results from graph theory are going to generalize to 3-hypergraphs?

Is it going to be more like graph theory or more like design theory?

First Off - Is there a universal friend 3-hypergraph? - And what does it look like?

It looks like a friendship 3-hypergraph with some vertex of the hypergraph that appears in a hyper-edge with each pair of vertices. Call this the universal friend 3-hypergraph.

Eg. For n=8.

∞01 ∞02 ∞03 ∞04 ∞05 ∞06 ∞12
∞13 ∞14 ∞15 ∞16 ∞23 ∞24 ∞25
∞26 ∞34 ∞35 ∞36 ∞45 ∞46 ∞56
013 124 235 346 450 561 602

Theorem(Sós) For $n = 2, 4 \mod 6$, there exists a universal friend 3-hypergraph.

Proof: For $n = 1, 3 \mod 6$, there exists a Steiner Triple System (STS). To the STS add the triples ∞ij where $0 \leq i \neq j \leq n-2$. Clearly, ∞ completes i, j, k as we have ∞ij, ∞ik and ∞jk. Also to find what completes ∞, i and j, find the triple containing ij. Say it is ijk. Then we have ∞ik, ∞jk and ijk. Because every pair ij occurs exactly once in an STS, there are no other completions.

Sós asked whether there were any other friendship 3-hypergraphs?

2 History

Hartke and Vandenbussche answered this in the affirmative. They formulated the problem as an integer programming problem. Using CPLEX to solve the integer program, they found Friendship 3-Hypergraphs on 8 vertices (unique), 16 vertices(≥ 3 non-isomorphic hypergraphs) and 32 vertices (≥ 1 non-isomorphic hypergraphs). The hypergraph was regular; i.e., all vertices appeared the same number of times. Further the 3 friendship graphs at 16 vertices had 208, 224 and 272 hyperedges (or triples).

Navin saw this and said these friendship graphs come from a geometry.

Theorem (H&V) Every edge must be contained in a unique K_4^3.

This means the triples of the friendship graph can be partitioned into K_4^3's or what we call quads or 4-sets. This makes it seem more design theory. We call this a friendship design with elements in quads.

eg. n=8 b=7

∞013 ∞124 ∞235 ∞346 ∞450 ∞561 ∞602

eg. n=8 b=8

0123 0145 0167 0246 1357 2345 2367 4567

or

0123 4567

0145 2367

0167 2345

0246 1357 planes of AG(2,3)

Think of each element in binary (low order bits on the right).

Let F be a field of order 2 and let V be a vector space of dimension $n+1$; i.e., $n+1$ tuples. There are 2^{n+1} vectors. $P(V)$ is the projective space of dimension n;.i.e., the lines of the vector space are the points of $P(V)$. There are $2^{n+1}-1$ points in $P(V)$. Let H be a hyperplane in $P(V)$. Then H has 2^n-1 points and dimension $n-1$. $A(V) = P(V)\backslash H$ is the affine space of 2^n points and dimension n. A plane in $P(V)$ has 7 points and a plane in $A(V)$ has 4 points. If the points in the affine plane are a, b, c, d then $a+b+c = d$. The other points in the hyperplane of $P(V)$ are $a+b$, $a+c$ and $b+c$. Each set of disjoint planes in $A(V)$ can be associated with the three points that form a line in the hyperplane $P(v)$ that do not occur on the hyperplane in $A(V)$ that goes through point 0. So instead of looking for sets of 2^{n-2} planes in $A(V)$, we will look for lines in the $P(V)$.

Now we want to translate this problem from finding sets of planes in the affine plane to finding lines in the projective plane.

Then we want a subset S of lines that obey two properties
1) There is no set of 4 lines in S such that 3 of them form a triangle and the 4'th line consists of the midpoints of the lines in the triangle. Property is symmetrical.

2) For any line, l, not in S there is a unique set of 3 lines in S such that l is a line through their midpoints.
Strange-like property

eg.Suppose we have lines 123, 145 and 246 in S. They form a triangle. Then we can not have line 356 in S as 356 is the line thru the midpoints of that triangle

1

3 5

2 6 4 this is property 1

Property 2) Now if the line 356 is not in S then there must be a unique set of 3 lines in S that form a triangle and such that 356 is the line thru the midpoints for each side. The 3 lines are not necessarily the ones shown.

0 1 2 3   4 5 6 7   8 9 10 11   12 13 14 15
0 1 4 5   2 3 6 7   8 9 12 13   10 11 14 15
0 1 6 7   2 3 4 5   8 9 14 15   10 11 12 13
:   :   :   :
0 7 11 13   1 6 10 13   2 5 9 14   3 4 8 15

instead of looking for the 68 4-sets above, we look for the 17 3-sets below

1 2 3,   1 4 5,   1 6 7,   1 14 15,   2 4 6,   2 12 14,   2 13 15,
3 8 11,   3 9 10,   4 10 14,   4 11 15,   5 8 13,   5 9 12
6 8 14,   6 9 15,   7 10 13,   7 11 12

So the recursion is 17 deep instead of 68 deep.

So we could search for a set S in the $P(V)$. A regular backtracking program could then easily search the state tree. We found a thousand or so solutions of which 3 were non-isomorphic on 13, 14 and 17 lines in the projective plane. They are equivalent to the ones found by H & V, but, in addition, we know there are no other friendship designs coming from the geometry.

Note the 17 lines become 68 quads which become 272 3-hyperedges.

At 5 tuples, the problem is too big too search completely and we only found the one that H & V found.

3 Bounds

Lemma 1 Every pair of elements occurs in at least one quad.

Lemma 2 There are at least $\lceil \frac{n(n-2)}{8} \rceil$ quads.

Proof: Start with H&V's idea of classifying hyperedges into three types:
1)those containing an element a. Denote the set of these by E_A
2)those that do not contain a but are contained in a K_4^3 with a, Denote the set of these by E^a and
3)the others that do not contain a, denote the set of these by E^B.

H & V proved that $|E^A| = |E_A|/3$. The key to getting a better lower bound is to carefully consider E^B.

E^B can be resolved into K_4^3's. There are $n(n-1)/2 - |E^A|$ pairs xy that do not occur with a and so must be in a K_4^3

We can only have 3 such pairs in a K_4^3.

Therefore there are at least $(n(n-1)/2 - |E^A|)/3$ K_4^3's in E^B.

Then some arithmetic finishes the proof.

Lemma 3 If a friendship hypergraph has $\frac{(n-1)(n-2)}{6}$ quads, then it is a universal friend hypergraph.

Proof: Pairs of points occur exactly 2 or $n-2$ times in such a hypergraph. From studying these we get the result.

Lemma 4 The number of quads is at most $n(n-1)^2/36$.

Proof: Count the number of quads that contain the pairs of a particular triple.

n	old lower bd $\lceil \frac{n(n-2)}{8} \rceil$	new lower bd $\lceil \frac{(n-1)(n-2)}{6} \rceil$	new upper bd $\frac{\binom{n}{3}(2n-6)}{4(3n-10)}$	old upper bd $\binom{n}{3}/4$	number of quads in design
4	1	1	1	1	1
5	2	2	2	3	-
6	3	4	4	4	-
7	5	5	7	6	-
8	6	7	10	14	7,8
9	8	10	14	21	-
10	10	12	21	30	12
11	13	15	28	41	-
12	15	19	38	55	-
13	18	22	49	71	?
14	21	26	62	91	26,?
15	25	31	78	113	?
16	28	35	95	140	35,52,56,68,?
32	120	155	836	1240	155,344,?

Table 1: Bounds and Number of Quads in Friendship Designs

One can see from the table that the number of quads in a friendship design is between a quadratic and a cubic.

Recently, I have looked at the bounds of friendship designs that come from a geometry.

I had to use integer and real programming to do it.

Unfortunately, the number of quads in even these more structured friendship designs is between a quadratic and a cubic - a slightly larger quadratic and slightly smaller cubic - very slightly

4 Our Computer Results

Running a straightforward backtracking program, we were able to reproduce all the complete searches that H&V did for $n \leq 10$. For $n = 11$, the straightforward approach is the Halting Problem.

The problem is that the search tree is very bushy. That is there are many choices from each node. Also we don't get to a dead end, until we are many levels down the search tree.

Our Algorithm - 7 Steps

1) For M= Max. # of pairs downto 2 do
the next 6 steps
We assume that that there is a pair that occurs M times and no pair occurs more than M times; i.e.,
n=10 M=4
1234
1256
1278
129t
We call this the starter set,

2)From the starter set, generate all possible sets that contain quads containing a 1 that does not cause two completions for some triple of elements. We will assume that these are the only quads in the set containing element 1. We check to see if every element occurs in a block with element 1.

eg. $n = 8, M = 3$
1234 1256 1278
1234 1256 1278 1357
1234 1256 1278 1358
1234 1256 1278 1357 1368
etc.

3)Eliminate isomorphic copies. Call what is left 1-sets

4)For each of the 1-sets generate a list of “forces”.

eg. $n = 9$ $M = 3$

1234

1256

1278

1357

Pair 13 occurs with elements 2,4,5,7

Pair 16 occurs with elements 2,5

Since there are no more quads with 1 in them, the only completion for 1,3,6 is the element 2. So the pair 36 must occur with a 2 in a quad. So the triple 236 must occur in some quad. The only possibilities are 2346, 2356, 2367, 2368 and 2369. But 2346 has a 3-intersection with the first quad, 2356 has a 3-intersection with the second quad. This leaves 2367 and 2368 and 2369 as forces. That is one of these 3 quads must be in the design. We get a list of these "forces" for a 1-set. If we can pick a quad from each "force", then we have a **candidate**. If not we have a dead end we can eliminate this possibility.

A 1-set may generate 0, 1, 2 or many candidates. All the candidates are grouped together.

We go through the "forces" in order of less dense to most dense. The whole idea is to reduce the bushiness of the search tree.

5)Eliminate isomorphic candidates.

6)Throw the candidates into a normal backtracking program that will see if they lead to solutions.

7)Eliminate isomorphic solutions.

Time in seconds

n \ M	2	3	4	5
6	0	.	.	.
7	0	.	.	.
8	0	0	.	.
9	0	0	.	.
10	3	14	1	.
11	66	86	11	.
12	1893	16310	17382	816

$n = 8$

Case \ M	2	3	4	5
All 1-sets	24	35	.	.
Non-isomorphic 1-sets	2	6	.	.
All Candidates	0	13	.	.
Non-isomorphic Candidates	0	5	.	.
Solutions	0	2	.	.
Non-isomorphic Solutions	0	2	.	.

$n = 9$

Case \ M	2	3	4	5
All 1-sets	157	482	.	.
Non-isomorphic 1-sets	4	14	.	.
All Candidates	0	6	.	.
Non-isomorphic Candidates	0	3	.	.
Solutions	0	0	.	.
Non-isomorphic Solutions	0	0	.	.

$n = 10$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	9	75	46	.
All Candidates	384	973	359	.
Non-isomorphic Candidates	4	131	108	.
Solutions	0	0	1	.
Non-isomorphic Solutions	0	0	1	.

$n = 11$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	12	461	1045	.
All Candidates	0	170514	18351	.
Non-isomorphic Candidates	0	11830	14504	.
Solutions	0	0	0	.
Non-isomorphic Solutions	0	0	0	.

$n = 12$

Case \ M	2	3	4	5
Non-iso. 1-sets	21	3414	39935	11410
All Candidates	0	71630269	76553127	3105440
Non-iso. Cand.	0	5825458	70819810	2802491
Solutions	0	0	0	0
Non-iso. Sol.	0	0	0	0

5 Conjectures

1)[H&V] There are no friendship 3-hypergraphs on an odd number of points.
2) There are no friendship 3-hypergraphs on $n = 0 \mod 6$ points.
3) No BIBD is a friendship design.
4) There are an infinite number of friendship 3-hypergraphs that are not universal friend 3-hypergraphs.
5) All friendship 3-hypergraphs are either a universal friend 3-hypergraph or are regular.
6) All non-universal friendship graphs are on 2^n points.

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