Lovász Local Lemma
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1 Introduction

In Alon [1] the following is proved using the Lovász Local Lemma.

Theorem 1.1 *If no symbol appears in more than* $(n-1)/(4e)$ *cells of an* $n \times n$ *array, then there is a transversal.*

A complete proof of this given along with a generalization of this theorem in the sequel.

2 Probability Theory

First we need to review some results from probability theory. We follow the outline by Ku [2]. By the definition of conditional probability, we have:

Proposition 2.1 $Pr(A|B) = \frac{Pr(A\cap B)}{Pr(B)}$.

Lemma 2.2 $Pr(A|B\cap C) = \frac{Pr(A\cap B|C)}{Pr(B|C)}$.

Proof From Proposition 2.1 we have:
$Pr(A|B\cap C) = \frac{Pr(A\cap B\cap C)}{Pr(B\cap C)}$. (*)
Rewriting Proposition 2.1 we have
$Pr(B\cap C) = Pr(B|C)Pr(C)$ and
$Pr(A\cap B\cap C) = Pr(A\cap B|C)Pr(C)$.
Substituting the above 2 equations into equation * gives us the answer. $\square$

Lemma 2.3 $Pr(\overline{A}\cap\overline{B}) = (1-Pr(A))(1-Pr(B|\overline{A}))$.

Proof By Proposition 2.1, we have
$Pr(\overline{B}\cap\overline{A}) = Pr(\overline{A})Pr(\overline{B}|\overline{A})$ or
$Pr(\overline{B}\cap\overline{A}) = (1-Pr(A)(1-Pr(B|\overline{A})$. $\square$

On iterating Lemma 2.3, we get

Corollary 2.4 $Pr(\cap_{i=1}^{n}\overline{A_i}) = \prod_{i=1}^{n}(1 - Pr(A_i| \cap_{j=1}^{i-1} \overline{A_j}))$.

Lemma 2.5 $Pr(\overline{A} \cap \overline{B}|C) = (1 - Pr(A|C))(1 - Pr(B|\overline{A} \cap C))$.

Proof By Proposition 2.1 we have

$$Pr(\overline{A} \cap \overline{B}|C) = \frac{Pr(\overline{B}\cap\overline{A}\cap C)}{Pr(\overline{C})} \text{ or}$$

$$Pr(\overline{A} \cap \overline{B}|C) = \frac{Pr(\overline{B}\cap\overline{A}\cap C)}{Pr(\overline{A}\cap C)}\frac{Pr(\overline{A}\cap C)}{Pr(C)}.$$

Again using Proposition 2.1 twice on the right hand side, we simplify the equation to

$$Pr(\overline{A} \cap \overline{B}|C) = Pr(\overline{B}|\overline{A} \cap C)Pr(\overline{A}|C) \text{ or}$$
$$Pr(\overline{A} \cap \overline{B}|C) = (1 - Pr(A|C))(1 - Pr(B|\overline{A} \cap C)).$$

□

This can be generalized to:

Corollary 2.6 $Pr(\cap_{i=1}^{n}\overline{A_i}|B) = \prod_{i=1}^{n}(1 - Pr(A_i| \cap_{j=1}^{i-1} \overline{A_j}|B))$.

3 Lovász Local Lemma or LLL

First we describe the most general version. Note the graph described in the lemma is not a dependency graph.

Theorem 3.1 {*Lovász Local Lemma-General Version-weak assumption*} *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Let* $G = (V, E)$ *be a graph on the vertex set* {*1,2,. . . ,n*}. *Suppose that there are real numbers* $x_1, x_2, \ldots, x_n$ *such that* $0 \le x_i < 1$ *for all* i *with the following holding for each* i*,*

$$Pr(A_i| \bigcap_{j \in S_2} \overline{A_j}) \le x_i \prod_{j \in S_1}(1 - x_j)$$

for all sets $S_2 \subseteq \{j : (i, j) \notin E\}$ *and where* $S_1 = \{j : (i, j) \in E\}$. *Then*

$$Pr(\bigcap_{i=1}^{n}\overline{A_i}) \ge \prod_{i=1}^{n}(1 - x_i).$$

Proof First we prove, by induction on t, that for any $T \subseteq \{1, 2, \ldots, n\}$, $|T| = t$ and $i \notin T$ that

$$Pr(A_i | \bigcap_{j \in T} \overline{A_j}) \leq x_i.$$

For $t = 0$, we have $Pr(A_i) \leq x_i \prod_{j \in S_2} (1 - x_j) \leq x_i$. So the hypothesis is true for $t = 0$. We will now assume the hypothesis is true for $0 \leq t' < t$ and prove the hypothesis true for t.

First, T is partitioned into T_1 and T_2 where $T_1 \subseteq \{j : (i, j) \in E\}$ and $T_2 = T \backslash T_1$. Then by Lemma 2.2

$$Pr(A_i | \bigcap_{j \in T} \overline{A_j}) = \frac{Pr(A_i \cap (\bigcap_{j_1 \in T_1} \overline{A_{j_1}} | \bigcap_{j_2 \in T_2} \overline{A_{j_2}})}{Pr(\bigcap_{j_1 \in T_1} \overline{A_{j_1}} | \bigcap_{j_2 \in T_2} \overline{A_{j_2}})}$$

The numerator is bounded above as follows:

$$Pr(A_i \cap (\bigcap_{j_1 \in T_1} \overline{A_{j_1}} | \bigcap_{j_2 \in T_2} \overline{A_{j_2}})) \leq Pr(A_i | \bigcap_{j_2 \in T_2} \overline{A_{j_2}})$$

as $Pr(A \cap B | C) \leq Pr(A|C)$. Then by our hypothesis in the theorem we have

$$Pr(A_i | \bigcap_{j_2 \in T_2} \overline{A_{j_2}}) \leq x_i \prod_{j \in T_1} (1 - x_j).$$

On the other hand the denominator can be bounded from below using the induction hypothesis. Suppose $T_1 = \{j_1, j_2, \ldots, j_r\}$. If $r = 0$ then the denominator is 1 and then

$$Pr(A_i | \bigcap_{j \in T_2} \overline{A_j}) \leq x_i \prod_{(i,j) \in E} (1 - x_j) \leq x_i$$

as required. So $r > 0$ and the denominator, using Corollary 2.6, becomes

$$\begin{aligned} &Pr(\overline{A_{j_1}} \cap \overline{A_{j_2}} \cap \ldots \cap \overline{A_{j_r}} | \cap_{k \in T_2} \overline{A_k}) \\ &= (1 - Pr(A_{j_1} | \cap_{k \in T_2} \overline{A_k}) \cdot (1 - Pr(A_{j_2} | \overline{A_{j_1}} \cap_{k \in T_2} \overline{A_k}) \cdots (1 - \\ &Pr(A_{j_r} | \overline{A_{j_1}} \cap \ldots \cap \overline{A_{j_{r-1}}} \cap_{k \in T_2} \overline{A_k})) \end{aligned}$$

which by the induction hypothesis becomes

$$\geq (1 - x_{j_1})(1 - x_{j_2}) \ldots (1 - x_{j_r}) \geq \prod_{j \in T_1} (1 - x_j).$$

So when we replace the numerator and the denominator by their bounds we get $Pr(A_i| \cap_{j \in T} \overline{A_j}) \leq x_i$. This completes the mathematical induction proof.

Now by Corollary 2.4, we get

$$Pr(\cap_{i=1}^{n}\overline{A_i}) = Pr(\overline{A_1})Pr(\overline{A_2}|\overline{A_1}) \cdots Pr(\overline{A_n}| \cap_{i=1}^{n-1} \overline{A_i})$$

$$= (1 - Pr(A_1))(1 - Pr(A_2|\overline{A_1}) \cdots (1 - Pr(A_n| \cap_{i=1}^{n-1} \overline{A_i})$$

$$\geq \prod_{i=1}^{n}(1 - x_i).$$

□

A version of LLL with a slightly stronger assumption is now proved as it is sometimes easier to use. The biggest change between the two versions is that this one assumes that that an event A_i is mutually independent of all the events A_j where vertex j is not joined to vertex i in the dependency graph.

Theorem 3.2 $\{$*Lovász Local Lemma-General Version-strong assumption*$\}$ *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Let* $G = (V, E)$ *be a graph on the vertex set {1,2,...,n}, called the dependency graph for the event* $A_1, A_2, \ldots, A_n$ *if for each* i, $1 \leq i \leq n$ *the event* A_i *is mutually independent of all events* $\{A_j : (i, j) \notin E\}$. *Suppose that there are real numbers* $x_1, x_2, \ldots, x_n$ *such that* $0 \leq x_i < 1$ *for all* i *with the following holding for each* i,

$$Pr(A_i) \leq x_i \prod_{j \in S_1}(1 - x_j),$$

where $S_1 = \{j : (i, j) \in E\}$. *Then*

$$Pr(\cap_{i=1}^{n}\overline{A_i}) \geq \prod_{i=1}^{n}(1 - x_i).$$

Proof Fix i. Let $S_2 \subseteq \{j : (i, j) \notin E\} = \{j_1, j_2, \ldots, j_s\}$. Then $A_i, A_{j_1}, \ldots, A_{j_s}$ are mutually independent, so

$$Pr(A_i| \bigcap_{j \in S_2} \overline{A_j}) = Pr(A_i) \leq x_i \prod_{(i,j) \in E}(1 - x_j).$$

The results now follows from Theorem 3.1.

□

Now we do the symmetric version of LLL. Note the graph is not a dependency graph.

Theorem 3.3 {*Lovász Local Lemma-Symmetric Version-weak assumption*} *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Let* $G = (V, E)$ *be a graph on the vertex set* {*1,2,...,n*} *such that no vertex is adjacent to more that d others. Suppose the following holds for each i,*

$$Pr(A_i | \bigcap_{j \in S_2} \overline{A_j}) \leq p,$$

for all sets $S_2 \subseteq \{j : ((i,j) \notin E\}$ *and if* $ep(d+1) \leq 1$, *then*

$$Pr(\bigcap_{i=1}^{n} \overline{A_i}) \geq \prod_{i=1}^{n} (1 - x_i).$$

Proof Assume $d > 0$. In Theorem 3.1, put $x_i = \frac{1}{1+d}$. Then we know that

$$\frac{1}{e} \leq (1 - \frac{1}{1+d})^d.$$

Multiplying through by ep we get

$$p \leq ep((1 - \frac{1}{1+d})^d.$$

Using our theorem's assumption we modify the above to

$$p \leq \frac{1}{1+d}((1 - \frac{1}{1+d})^d.$$

which is equivalent to

$$p \leq x_i \prod_{(i,j) \in E} (1 - x_j).$$

So the assumptions of Theorem 3.1 holds and so does its result. □

The strong assumption version of the symmetric LLL follows just as easily.

Theorem 3.4 {*Lovász Local Lemma-Symmetric Version-strong assumption*} *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Let* $G = (V, E)$ *be a graph on the vertex set* {*1,2,...,n*}*, called the dependency graph for the* *event* $A_1, A_2, \ldots, A_n$ *if for each* i*,* $1 \leq i \leq n$ *the event* A_i *is mutually independent of all events* $\{A_j : (i,j) \notin E\}$ *except for at most* d *of them. If for all* i*,*

$$Pr(A_i) \leq p$$

Then

$$Pr(\bigcap_{i=1}^{n} \overline{A_i}) \geq \prod_{i=1}^{n} \frac{1}{(1+d)}.$$

4 Application to transversals in rectangles.

Let $A = (a_i)$ be an $n \times n$ array with integer entries from $\{1, 2, \ldots, s\}$. The application does not use a dependency graph so Theorem 3.3 will be used.

Theorem 4.1 *If no symbol appears in more than* $k \leq (n-1)/(4e)$ *cells of an* $n \times n$ *array,* A*, then there is a transversal in* A*.*

Proof Let π be a random permutation of $\{1, 2, \ldots, n\}$, chosen from a uniform distribution among all possible $n!$ permutations. Denote by T the set of all ordered fourtuples (i, j, i', j') satisfying $i < i'$ and $j \neq j'$ and $a_{ij} = a_{i'j'}$. For each $(i, j, i', j') \in T$, let $A_{iji'j'}$ denote the event that $\pi(i) = j$ and $\pi(i') = j'$. The existence of a transversal in A is equivalent to the statement that $Pr(\bigcap_{iji'j' \in T} \overline{A}_{iji'j'}) > 0$. Define a graph $G = (V, E)$ on the vertex set T by making (i, j, i', j') adjacent to (p, q, p', q') if and only if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$, i.e.; if and only if the fourtuples occupy 4 different rows and 4 different columns. The maximum degree in G is less than $(4n-4)(k-1) \leq 4nk$ as there are 2 rows of length n and 2 columns of length n for the the adjacency to occur in but 4 cells are counted twice. There are at most $k-1$ other elements the same as the adjacent element. Since $e \cdot 4nk \cdot \frac{1}{n(n-1)} < 1$ the result will follow by Theorem 3.3, if for each (i, j, i', j'),

(***) $Pr(A_{iji'j'} | \cap_S \overline{A}_{pqp'q'}) \leq \frac{1}{n(n-1)}$, for any set

$S \subseteq \{(p, q, p', q') : ((p, q, p', q,), (i, j, i', j')) \notin E\}$

To prove (***), we just need to prove it for $i = j = 1$ and $i' = j' = 2$. Symmetry then will apply it to all i, j, i', j'. We define a *good* permutation to be a section that has no duplicate entries for duplicate entries not adjacent to (1,1,2,2). So the permutation restricted to rows and columns $3, 4, \ldots, n$

is a transversal. Now define S_{ij} to be the set of good permutations that go through $(1, i)$ and $(2, j)$, i.e. the good permutations, π satisfying $\pi(1) = i$ and $\pi(2) = j$.

We claim that $|S_{12}| \leq |S_{ij}|$ for all $i \neq j$. We prove this claim by showing an injective (every element of the domain is mapped to a distinct element of the range) function from the set of good permutations in S_{12} to the set of good permutations in S_{ij}. There are five cases.

First, let $\{1, 2\} \cap \{i, j\} = \emptyset$. For any $\pi \in S_{12}$, we define π^* as follows. Suppose $\pi(x) = i$ and $\pi(y) = j$, then $\pi^*(1) = i$, $\pi^*(2) = j$, $\pi^*(x) = 1$, $\pi^*(y) = 2$ and $\pi^*(t) = \pi(t)$ for all $t \neq 1, 2, x, y$. Now π^* is good because π is good. They only differ in four cells but they involve row and columns 1 and 2 and so can not effect the goodness of the permutation. So $|S_{12}| \leq |S_{ij}|$.

Second, let $i = 1$ and $j \neq 1$ or 2. Suppose $\pi(y) = j$, then $\pi^*(1) = 1$, $\pi^*(2) = j$, $\pi^*(y) = 2$ and $\pi^*(t) = \pi(t)$ for all $t \neq 1, 2, x, y$. Just as before π^* is good. So $|S_{12}| \leq |S_{1j}|$.

Third, let $j = 2$ and $i \neq 1$ or 2. Same proof as above so $|S_{12}| \leq |S_{i2}|$.

Fourth, let $i = 2$ and $j = 1$. Then $\pi^*(1) = 2$, $\pi^*(2) = 1$. and π^* is obviously good and So $|S_{12}| \leq |S_{21}|$.

Fifth, let $i = 1$ and $j = 2$. Obviously $|S_{12}| \leq |S_{12}$.

So $|S_{12} \leq |S_{ij}|$ for any i, j where $i \neq j$. Letting i and j take on values from $1, 2 \ldots n$ gives us $n(n-1)$ inequalities. So, by adding these inequalities we get

$$n(n-1)S_{12} \leq \sum_{i \neq j} S_{ij}$$

But $\sum_{i \neq j} S_{ij}$ counts all the good permutations that pairs of identical elements which do not involve row and columns 1 and 2. But then $\sum_{i \neq j} S_{ij}/n! = Pr(\cap_S \overline{A_{pqp'q'}})$. On the other hand $S_{12}/n! = Pr(A_{1122} \cap_S \overline{A_{pqp'q'}})$. Substituting these two values into the above displayed inequality gives

$$\begin{aligned}
Pr(A_{1122}|\bigcap_S \overline{A_{pqp'q'}}) &= \frac{Pr(A_{1122}\bigcap_S \overline{A_{pqp'q'}})}{Pr(\bigcap_S \overline{A_{pqp'q'}})} \\
&= \frac{S_{12}/n!}{\sum_{i\neq j} S_{ij}/n!} \\
&\leq \frac{1}{n(n-1)}
\end{aligned}$$

This proves (***).

□

We give an example. The squares with a dot contain some integer we are not interested in.

$$\begin{pmatrix} 1 & . & 6 & 8 & . \\ . & 1 & . & . & 8 \\ . & . & 2 & 3 & 4 \\ 7 & . & 4 & 2 & 3 \\ . & 7 & 5 & 5 & 6 \end{pmatrix}$$

We are considering A_{1122}. $S = \{(3344), (3445), (3543)\}$.

S_{12} contains 3 good permutations which we find by searching:
π_1=(11)(22)(34)(43)(55)
π_2=(11)(22)(33)(45)(54)
π_3=(11)(22)(35)(44)(53)
S_{35} must contain the following 3 good permutations constructed from the 3 good permutations in S_{12}. This is case 1.
π_1^*=(13)(25)(34)(41)(52)
π_2^*=(13)(25)(31)(42)(54)
π_3^*=(13)(25)(32)(44)(51)
Note that $\pi *_1$ has a pair of identical entries in it but the cells containing the pair are in the first and second columns so it does not disqualify $\pi *_1$ from being good. Note that $\pi *_i$ has only one cell in the bottom right 3 by 3 subsquare, so there is no way $\pi *_i$ could not be good in this trivial example. Actually $|S_{35}| = 6$.

In S_{13} the good permutation that corresponds to the first good permutation in S_{12} is (11)(23)(34)(42)(55). Here there are two positions in S_{13} left over from S_{12} after the function has been applied. But if they are good in the one they must be good in the other. $|S_{13}| = 5$.

This theorem can be generalized quite easily. Let $A = (a_{ij})$ be an $m \times n$ array with integer entries from $\{1, 2, \ldots, s\}$. *A transversal in an* $m\times n$ *array,*

A, is a set of cells from A, exactly one from each row and at most one from each column of A.

Theorem 4.2 *If no symbol appears in more than* $\frac{n-1}{2e(1+m/n)}$ *cells of an* $m \times n$ *array,* A*, then there is a transversal in* A*.*

Proof Let π be a random permutation of m of the elements from $\{1, 2, \ldots, n\}$, chosen from a uniform distribution among all possible $n!(n-1)! \cdots (n-m+1)$ such permutations. Denote by T the set of all ordered fourtuples (i, i, i', j') satisfying $i < i'$ and $j \neq j'$ and $a_{ij} = a_{i'j'}$. For each $(i, j, i', j') \in T$, let $A_{iji'j'}$ denote the event that $\pi(i) = j$ and $\pi(i') = j'$. The existence of a transversal in A is equivalent to the statement that $Pr(\bigcap_{iji'j' \in T} \overline{A}_{iji'j'}) > 0$. Define a graph $G = (V, E)$ on the vertex set T by making (i, j, i', j') adjacent to (p, q, p', q') if and only if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$, i.e.; if and only if the fourtuples occupy 4 different rows and 4 different columns. The maximum degree in G is less than $(2m + 2n - 4)(k - 1) \leq 2(m + n)k$ as there are 2 rows of length n and 2 columns of length m for the the adjacency to occur in but 4 cells are counted twice. There are at most $k - 1$ other elements the same as the adjacent element. Since $e \cdot 2(m + n)k \cdot \frac{1}{n(n-1)} \leq 1$ the result will follow by Theorem 3.3, if for each (i, j, i', j'),

($\sharp$) $Pr(A_{iji'j'} | \cap_S \overline{A}_{pqp'q'}) \leq \frac{1}{n(n-1)}$,for any set

$S \subseteq \{(p, q, p', q') : ((p, q, p', q,), (i, j, i', j')) \notin E\}$

To prove ($\sharp$), we just need to prove it for $i = j = 1$ and $i' = j' = 2$. Symmetry then will apply it to all i, j, i', j'. We define a *good* permutation to be a section that has no duplicate entries for duplicate entries not adjacent to (1,1,2,2). So the permutation restricted to rows and columns that are not 1 or 2 is a transversal. Now define S_{ij} to be the good permutations that go through $(1, i)$ and $(2, j)$, i.e. the good permutations, π satisfying $\pi(1) = i$ and $\pi(2) = j$.

We claim that $|S_{12}| \leq |S_{ij}|$ for all $i \neq j$. We prove this claim by showing an injective (every element of the domain is mapped to a distinct element of the range) function from the set of good permutations in S_{12} to the set of good permutations in S_{ij}. The proof is the same as in the previous proof except that sometimes $\pi^{-1}(i)$ and $\pi^{-1}(j)$ do not exist. In those cases $\pi^*(i) = \pi(i)$.

So $S_{12} \leq S_{ij}$ for any i, j where $i \neq j$. Adding up all cases gives

$$n(n-1)S_{12} \leq \sum_{i \neq j} S_{ij}$$

But $\sum_{i \neq j} S_{ij}$ counts all the good permutations.
But then $\sum_{i \neq j} S_{ij} / \prod_{i=1}^{m} (n-i+1) = Pr(\cap_S \overline{A_{pqp'q'}})$. On the other hand $S_{12} / \prod_{i=1}^{m} (n-i+1) = Pr(A_{1122} \cap_S \overline{A_{pqp'q'}})$. Substituting these two values into the above displayed inequality gives

$$Pr(A_{1122} | \bigcap_S \overline{A_{pqp'q'}}) = \frac{Pr(A_{1122} \bigcap_S \overline{A_{pqp'q'}})}{Pr(\bigcap_S \overline{A_{pqp'q'}})} \leq \frac{1}{n(n-1)}$$

This proves (♯).

□

References

[1] N. Alon, Probabilistic proofs of existence of rare events, Springer Lecture Notes in Mathematics No. 1376 (J. Lindenstrauss and V. D. Milman Eds.), Springer-Verlag (1988), 186-201

[2] C. Y. Ku, Lovász Local Lemma, internet article, http://www.maths.qmul.ac.uk/∼cyk/local.pdf