THE INVESTIGATION OF THE STRUCTURE OF A (22, 33, 12, 8, 4) DESIGN.

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INTRODUCTION

There are many parameter sets for which it is not known whether there exists a balanced incomplete block design or not. The parameter set (22,33,12,8,4) is one of them. By examining the structure of the intersection between blocks, it is hoped that either non-existence can be proved or enough of the structure of the design will be determined, so that the design itself can be constructed. This essay is an attempt to reach this goal.

GENERAL RESULTS

Balanced Incomplete Block Designs

A balanced incomplete block design is an arrangement of v distinct elements or varieties, into b subsets or blocks, such that the following conditions are met:

(1.1.1) Each block contains exactly k distinct varieties.

(1.1.2) Each variety occurs in exactly r different blocks.

(1.1.3) Each pair of distinct varieties occurs in exactly λ blocks.

In order to exclude trivial designs the following are added:

(1.1.4) $\lambda > 0$

(1.1.5) $r > \lambda$

From the above definition, we note there are five parameters for a balanced incomplete block design. In what follows, the parameters will be written in the standard order (v,b,r,k,λ) and the word "design" will stand for the phrase "balanced incomplete block design".

There are two basic relations on the five parameters for a design.

(1.1.6) $bk = rv$

(1.1.7) $r(k-1) = \lambda(v-1)$.

The first relation counts the total number of times all varieties appear in the design, each of the b blocks containing k varieties and each of v varieties being contained in r blocks. The second relation counts the occurrences of pairs containing a particular variety, say x. The variety x occurs in r blocks and in each of them, it pairs with the $(k-1)$ remaining varieties, while on the other hand, x is paired λ times with each of the remaining $(v-1)$ varieties.

Since the above argument is true for any variety, it can be noted that (1.1.1) and (1.1.3) imply that (1.1.2) is redundant. Also note that relations (1.1.6) and (1.1.7) are necessary conditions for the existence of a balanced incomplete block design.

Fisher's Inequality

Fisher [2] discovered another necessary condition for the existence of designs. The proof of the theorem will use the concept of the incidence matrix, although Fisher proved the theorem another way.

Let the blocks of the design be $B_1, B_2, \ldots, B_b$ and let the varieties be $a_1, a_2, \ldots, a_v$.

Definition: The matrix $A = (a_{ij})$ $i = 1,\ldots,v$, $j = 1,\ldots,b$ where

$$a_{ij} = 1 \quad \text{if} \quad a_i \in B_j$$

$$a_{ij} = 0 \quad \text{if} \quad a_i \notin B_j$$

is called the incidence matrix of the design.

THEOREM 1.2.1. In a balanced incomplete block design with parameters (v,b,r,k,λ), it is necessary that

(1.2.2) $\quad b \geq v.$

Proof: Suppose that $b < v$. Let $(v-b)$ columns of zeros be appended to A to form the $v \times v$ matrix A_1.

The basic requirements of a design, leads to the following result:

$$AA^T = \begin{bmatrix} r & \lambda & \cdot & \cdot & \cdot & \lambda \\ \lambda & r & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & & \lambda \\ \lambda & \cdot & \cdot & \cdot & \lambda & r \end{bmatrix}$$

Now $AA^T = A_1A_1^T$ because the appended zeros do not change the value of inner products of the rows. Thus $|AA^T| = |A_1A_1^T| = |A_1||A_1^T|$. But $|A_1| = 0$ as it has a zero column. Thus $|AA^T| = 0$. $|AA^T|$ can be evaluated directly by

1. Subtract the first column from columns 2, 3, ..., v.
2. Add to the first row, rows 2, 3, ..., v.
3. Evaluate by expanding along the first row

$$|AA^T| = [r + (v-1)\lambda]\ (r-\lambda)^{v-1}$$

$$|AA^T| = rk(r-\lambda)^{v-1}.$$

Since r, k and $(r-\lambda)$ are all non-zero, $|AA^T|$ is non-zero. This is a contradiction and the theorem follows.

Block Intersection Numbers

Although the balanced incomplete block design with parameters (22,33,12,8,4) satisfies conditions (1.1.6), (1.1.7) and (1.2.2) this does not imply that the design exists. Therefore the structure of designs will now be examined.

Consider a design with parameters (v,b,r,k,λ) and consider a particular block say B_1. Let a_i equal the number of blocks in the design which have exactly i varieties in common with B_1.

Definition: The numbers a_i $(i=0,1,2,\ldots,k)$ are called block intersection numbers for the block B_1. If $i > k$, then $a_i = 0$.

THEOREM 1.3.1. Consider a balanced incomplete block design with parameters $(v,b,k+\lambda,k,\lambda)$. Let B_1 be a particular block in the design and let the other blocks be labelled $B_2,B_3,\ldots,B_b$ then:

(1.3.2) $$\sum_{i=0}^{k} a_i = b-1$$

(1.3.3) $$\sum_{i=0}^{k} ia_i = k(r-1)$$

(1.3.4) $$\sum_{i=0}^{k} i^2 a_i = k^2\lambda .$$

Proof: Since a_i counts the number of blocks intersecting with B_1 in i varieties, the left hand side of (1.3.2) counts all blocks other than B_1. Thus (1.3.2) follows.

Similarly ia_i counts the $(r-1)$ occurrences of the k varieties of B_1, in the blocks $B_2, B_3, \ldots, B_b$, which intersect B_1 in i varieties. Thus the left hand side of (1.3.3) counts all these occurrences in $B_2, B_3, \ldots, B_b$ and (1.3.3) follows.

Consider

(1.3.5) $$\sum_{i=0}^{k} \binom{i}{2} a_i = \binom{k}{2}(\lambda-1) .$$

In the same way $\binom{i}{2}a_i$ counts all $(\lambda-1)$ occurrences of all pairs of elements B_1, in the blocks $B_2, B_3, \ldots, B_b$ which intersect B_1 in i varieties. Thus the left hand side of (1.3.5) counts these occurrences in all of $B_2, B_3, \ldots, B_b$ and (1.3.5) follows.

Then

$$\sum_{i=0}^{k} \binom{i}{2} a_i = \binom{k}{2}(\lambda-1)$$

$$\sum_{i=0}^{k} i(i-1)a_i = k(k-1)(\lambda-1)$$

$$\sum_{i=0}^{k} i(i-1)a_i + \sum_{i=0}^{k} ia_i = k(k-1)(\lambda-1)+k(r-1)$$

$$\sum_{i=0}^{k} i^2 a_i = k(k\lambda-k-\lambda+r)$$

But $r = \lambda + k$

$$\sum_{i=0}^{k} i^2 a_i = k^2\lambda \quad \square.$$

Connor's Criterion

Definition: Consider a design with parameters (v,b,r,k,λ) and consider its $v\times b$ incidence matrix A. Then

$$A^T A = (S_{ij}) = S$$

is the structural matrix of the design.

Clearly S_{ij} is the number of rows of A which have 1's in both column i and column j; that is, S_{ij} is the number of varieties common to block i and block j. The cardinality of this intersection will also be denoted by $n(B_i \cap B_j)$.

Definition: If t blocks of the design are considered, say $B_1, B_2, \ldots, B_t$, then let the $v\times t$ submatrix of A which corresponds to $B_1, B_2, \ldots, B_t$ be A_t. Then the $t\times t$ symmetric matrix

$$S_t = A_t^T A_t$$

is called the structural matrix of the t chosen blocks.

Definition: Suppose we have a set of t blocks with structural matrix S_t. Then

$$C_t = r(r-\lambda)I_t + \lambda k J_t - rS_t$$

is called the characteristic matrix of the t chosen blocks.

Note

$$C_{jj} = (r-k)(r-\lambda)$$
(1.4.1)
$$C_{ij} = \lambda k - rS_{ij} \qquad (i \neq j) .$$

A theorem due to Connor [1] will now be proved. The proof in this essay is a slight particularization of Connor's own proof.

THEOREM 1.4.2. If C_t is the characteristic matrix of any set of t blocks chosen from a balanced incomplete block design with parameters (v,b,r,k,λ) then

(1.4.3) $|C_t| \geq 0$, if $t \leq b - v$

(1.4.4) $|C_t| = 0$, if $t > b-v$

(1.4.5) $kr^{-b+v+1}(r-\lambda)^{2v-b-1}|C_{b-v}|$ is a perfect integral square.

Proof:

Let
$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ I_t & 0 \end{bmatrix}$$

where $A = (A_{11}, A_{12})$ i.e. the incidence matrix

$A_{11} = (a_{ij})$ $i = 1,\ldots,v$ $j = 1,\ldots,t$

$A_{12} = (a_{ij})$ $i = 1,\ldots,v$ $j = t+1,\ldots,b.$

Then
$$A_1A_1^T = \begin{bmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{11} \\ A_{11}^T & I_t \end{bmatrix}$$

$$A_1A_1^T = \begin{bmatrix} B & A_{11} \\ A_{11}^T & I_t \end{bmatrix}$$

where $B = (r-\lambda)I_v + \lambda J_v$.

To evaluate $|A_1A_1^T|$,

1. Multiply the last t columns by $rk(r-\lambda)$ and write an offsetting factor outside.
2. Add rows $1,2,\dots,v-1$ to row v.
3. Take the factor rk out of row v.
4. Multiply row v by λ and subtract this product from rows $1,2,\dots,v-1$.
5. Take the factor $r-\lambda$ out of rows $1,2,\dots,v-1$.
6. Subtract rows $1,2,\dots,v-1$ from row v.
7. Subtract suitable multiples of columns $1,2,\dots,v$ from column $v+1, v+2,\dots,v+t$ so as to make the elements that are both in the first v rows and the last t columns equal to zero.

$$|A_1A_1^T| = k\, r^{-t+1}(r-\lambda)^{v-t-1}|C_t|.$$

From the theory of quadratic forms, it follows that for a real matrix A_1, the $|A_1A_1^T| \geq 0$. Since k, r, $(r-\lambda)$ are positive $|C_t| \geq 0$. Also the rank $(A_1A_1^T) \leq$ rank A_1, whence if $t > b - v$ then $A_1A_1^T$ is singular and thus $|C_t| = 0$. If $t = b - v$, then A_1 is a square matrix and $|A_1A_1^T| = |A_1|^2$ and (1.4.5) follows.

Majumdar's Result

Majumdar [6] proved that two blocks of a balanced incomplete block design intersect in $(k+\lambda-r)$ varieties if and only if they intersect every other block of the design in an equal number of varieties. His proof uses counting arguments whereas Majumdar's Theorem, presented here, uses Connor's Criterion. Majumdar's proof is more elegant and shorter.

THEOREM 1.5.1. In a balanced incomplete block design with parameters (v,b,r,k,λ), if two blocks intersect in $(k+\lambda-r)$ varieties then these two blocks intersect every other block in an equal number of varieties.

Proof: If $r - k = 0$ we have a symmetric design and it is well known [3] that every block intersects every other block in λ varieties. This validates the theorem for $r - k = 0$.

Let $r - k > 0$ and let two blocks, say B_1 and B_2, intersect in $(k+\lambda-r)$ varieties. Let B_3 be any other third block of the design. Now the characteristic matrix can be filled in.

$$\begin{aligned} c_{12} = c_{21} &= \lambda k - rs_{12} \\ &= \lambda k - r(k+\lambda-r) \\ &= (r-k)(r-\lambda) \end{aligned}$$

$$C = \begin{bmatrix} (r-k)(r-\lambda) & (r-k)(r-\lambda) & \lambda k-rc_1 \\ (r-k)(r-\lambda) & (r-k)(r-\lambda) & \lambda k-rc_2 \\ \lambda k-rc_1 & \lambda k-rc_2 & (r-k)(r-\lambda) \end{bmatrix}$$

where $c_1 = n(B_1 \cap B_3)$, $c_2 = n(B_2 \cap B_3)$

$$|C| = (r-\lambda)(r-k)[2(\lambda k-rc_1)(\lambda k-rc_2)-(\lambda k-rc_1)^2 - (\lambda k-rc_2)^2]$$

$$|C| = -r^2(r-\lambda)(r-k)[c_1{}^2 + c_2{}^2 - 2c_1c_2].$$

If $c_1 \neq c_2$ then $c_1{}^2 + c_2{}^2 - 2c_1c_2 > 0$ and since r, $r-\lambda$, $r-k$ are positive then $|c| < 0$. But (1.4.3) implies $|c| \geq 0$. This is a contradiction. Thus $c_1 = c_2$. Since B_3 was any third block the theorem follows.

RESULTS ON (22, 33, 12, 8, 4) DESIGN

Classes of Blocks

Since (1.1.6), (1.1.7) and (1.2.2) do not rule out a (22, 33, 12, 8, 4) design, the theorems of the first part of the essay will be applied to this design.

Applying relations (1.3.2), (1.3.3) and (1.3.4), the following results are obtained.

(2.1.1) $$\sum_{i=0}^{8} a_i = 32$$

(2.1.2) $$\sum_{i=0}^{8} ia_i = 88$$

(2.1.3) $$\sum_{i=0}^{8} i^2 a_i = 256.$$

The result of multiplying (2.1.1), (2.1.2) and (2.1.3) by 6, -5 and 1, respectively and adding is

$$\sum_{i=0}^{8} (i-3)(i-2)a_i = 8$$

which is

(2.1.4) $$6a_0 + 2a_1 + 2a_4 + 6a_5 + 12a_6 + 20a_7 + 30a_8 = 8.$$

(2.1.4) and (2.1.1) can now be used to prove:

THEOREM 2.1.5. In a balanced incomplete block design with parameters (22, 33, 12, 8, 4) the block intersection number $[a_i]$ of any block are given by one of the nine cases below ($a_i = 0$ for $i \geq 6$).

Class	a_0	a_1	a_2	a_3	a_4	a_5
I	0	1	8	22	0	1
II	0	0	11	19	1	1
III	1	1	3	27	0	0
IV	1	0	6	24	1	0
V	0	4	0	28	0	0
VI	0	3	3	25	1	0
VII	0	2	6	22	2	0
VIII	0	1	9	19	3	0
IX	0	0	12	16	4	0

<u>The Non-Existence of Blocks of Class III and V</u>

Suppose a block of class III exists in some design D. Then there exists two blocks of D which intersect in one variety. In general, B_1 and B_2 can be represented as follows:

$$B_1 = \{\theta, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$$

$$B_2 = \{\theta, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$$

$$a_i \neq b_j \quad \text{for all } i,j.$$

There are 7, θa_i pairs in B_1. They, each, must occur three more times in the design. Thus there are 21, θa_i pairs in $B_3, B_4, \ldots, B_{33}$.

θ can occur with at most 2 a_i's in the remaining blocks since $a_j = 0$, $j = 4,5,6,7,8$. Since $r = 12$, pairs of the form θa_i can occur at most, 20 times in the remaining blocks. Therefore the design can not exist.

The proof for class V blocks is exactly the same so the following theorem can be stated.

THEOREM 2.2.1. In a balanced incomplete block design with parameters (22, 33, 12, 8, 4), these can be no blocks of class III or class V.

The Non-Existence of Blocks of Class I

Suppose a block of class I existed in some design D with parameters (22, 33, 12, 8, 4). Then there exists two blocks, say B_1 and B_2, in D which intersects in 5 varieties. In general B_1 and B_2 can be represented by

$$B_1 = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, a_1, a_2, a_3\}$$

$$B_2 = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, b_1, b_2, b_3\}$$

$$a_i \neq b_j \quad \text{for all } i,j.$$

Now, since B_1 is a class I block, there exists a block, say B_3, which intersects B_1 in one variety. Let $n(B_2 \cap B_3) = s_{32}$.

Since B_2 has $a_5 = 1$, B_2 must be a block of class I or class II. Therefore $s_{32} = 1, 2, 3$ or 4. The characteristic matrix can now be filled in using (1.4.1)

$$C_{jj} = (r-k)(r-\lambda) \qquad C_{12} = C_{21} = \lambda k - rs_{21}$$

$$C_{jj} = 32 \qquad C_{12} = C_{21} = 32 - 12(5)$$

$$C_{12} = C_{21} = -28$$

$$C_{13} = C_{31} = \lambda k - rs_{31} \qquad C_{23} = C_{32} = \lambda k - rs_{32}$$

$$C_{31} = 32 - 12(1) \qquad C_{32} = 32 - 12s_{32}$$

$$C_{31} = 20 \qquad C_{32} = 20,\ 8,\ -4,\ \text{or } -16.$$

$$C = \begin{bmatrix} 32 & -28 & 20 \\ -28 & 32 & C_{23} \\ 20 & C_{32} & 32 \end{bmatrix}$$

if $C_{32} = 20$	$C = -40320$
if $C_{32} = 8$	$C = -16128$
if $C_{32} = -4$	$C = -1152$
if $C_{32} = -16$	$C = +4608$

Using Connor's Criterion (1.4.3), it can be concluded that $n(B_2 \cap B_3) = 4$. Thus B_2 is a class II block as class I blocks have $a_4 = 0$.

If it can now be proved that a class I block can not intersect a class II in five varieties, then a class I block can not exist.

Consider the remaining blocks of the design D as they intersect B_1 and B_2. Since $n(B_1 \cap B_3) = 1$ and $n(B_2 \cap B_3) = 4$, B_3 can be represented, in general, as follows:

$$B_3 = \{\theta_\ell, b_1, b_2, b_3, c_1, c_2, c_3, c_4\}$$

$$a_i \neq c_j \neq b_k \text{ for all } i,j,k.$$

Now blocks B_4 to B_{33} must intersect B_1 and B_2 in either 2 or 3 varieties. Therefore the blocks may be classified as follows:

Type 1: B_i is of Type 1 if it contains two varieties from each of B_1 and B_2.

Type 2: B_i is of Type 2 if it contains two varieties from one of B_1 or B_2, and three varieties from the other.

Type 3: B_i is of Type 3 if it contains three varieties from each of B_1 and B_2.

Each of the above three types can be divided into distinct sub-types as shown below. The sub-type of a block B_i depends on the number of θ_i elements it contains. Note that a type 1 block can contain either one, two or none of the θ_i (corresponding to types 11,12,13 below). The following are the 10 block types.

Type 11: B_i contains $\theta_i\theta_j$.

Type 12: B_i contains $\theta_i a_j b_k$.

Type 13: B_i contains $a_i a_j b_k b_\ell$.

Type 21: B_i contains $\theta_i\theta_j a_k$ or $\theta_i\theta_j b_k$.

Type 22: B_i contains $\theta_i a_j a_k b_\ell$ or $\theta_i a_j b_k b_\ell$.

Type 23: B_i contains $a_i a_j a_k b_\ell b_m$ or $a_i a_j b_k b_\ell b_m$.

Type 31: B_i contains $\theta_i\theta_j\theta_k$.

Type 32: B_i contains $\theta_i\theta_j a_k b_\ell$.

Type 33: B_i contains $\theta_i a_j a_k b_\ell b_m$.

Type 34: B_i contains $a_i a_j a_k b_\ell b_m b_n$.

Seven equations, counting the total number of varieties or variety pairs can now be written down.

(2.3.1) $x_{12} + 4x_{13} + 2x_{22} + 6x_{23} + x_{32} + 4x_{33} + 9x_{34} = 36$

(2.3.2) $2x_{13} + x_{22} + 4x_{23} + 2x_{33} + 6x_{34} = 15$

(2.3.3) $2x_{12} + 2x_{21} + 3x_{22} + \quad + 4x_{32} + 4x_{33} = 87$

(2.3.4) $x_{11} + x_{21} + 3x_{31} + x_{32} = 20$

(2.3.5) $2x_{11} + x_{12} + 2x_{21} + x_{22} + 3x_{31} + 2x_{32} + x_{33} = 49$

(2.3.6) $2x_{12} + 4x_{13} + x_{21} + 3x_{22} + 5x_{23} + 2x_{32} + 4x_{33} + 6x_{34} = 63$

(2.3.7) $x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} + x_{34} = 30$

Note that x_{ij} is the number of blocks (from B_4 to B_{33}) of type ij. Thus (2.3.1) counts the number of $a_i b_j$ pairs that occur in B_4 to B_{33}. Since there are 9 such pairs which have to occur 4 times each and since they do not occur in B_1, B_2, or B_3, the total number of such pairs in 36.

Similarly (2.3.2) counts $a_i a_j$ pairs plus $b_i b_j$ pairs in B_4 to B_{33}. Since there are 6 such pairs which must occur 4 times each and since 9 such pairs occur in B_1, B_2 and B_3, the total sum of such pairs, remaining is 15. Again (2.3.3) counts $\theta_i a_j$ pairs plus $\theta_i b_j$ pairs and (2.3.4) counts $\theta_i \theta_j$ pairs.

Now (2.3.6) counts the total occurrences of a's and b's in B_4 to B_{33}. There are 6 such varieties which must occur 12 times each and they occur 9 times in B_1, B_2 and B_3, then there are 63 of those varieties in B_4 to B_{33}. Similarly (2.3.5) counts θ_i elements and (2.3.7) counts the number of blocks.

Now if (2.3.1), (2.3.2) and (2.3.6) are multiplied by -2, 2 and 1, respectively and if then they are summed, the result is:

(2.3.8) $$x_{21} + x_{22} + x_{23} = 21.$$

But the number of blocks that intersect B_1 in 2 varieties is 8 while the number that intersect B_2 in 2 varieties is 11. Thus

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \leqq 8 + 11$$

$$\therefore \quad x_{21} + x_{22} + x_{23} \leq 19.$$

This is a contradiction with (2.3.8) and thus the following theorem can be stated.

THEOREM 2.3.9. In a balanced incomplete block design with parameters (22,33,12,8,4) there can be no blocks of class I.

Non-Existence of Blocks of Class II

The proof here follows the pattern of the previous proof quite closely.

Assume a class II block exists in a design D with parameters (22,33,12,8,4). There must exist two blocks, say B_1 and B_2, which intersect in 5 varieties and can be represented as follows:

$$B_1 = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, a_1, a_2, a_3\}$$

$$B_2 = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, b_1, b_2, b_3\} \quad a_i \neq b_j \text{ for all } i,j.$$

Now another block, say B_3, exists which intersects B_1 in four varieties as B_1 has $a_4 = 1$. Let $n(B_2 \cap B_3) = s_{32}$.

Since B_2 has $a_5 = 1$ it must be a class II block so $s_{32} = 2$, 3 or 4. Thus the characteristic matrix can now be filled in using (1.4.1).

$$C = \begin{bmatrix} 32 & -28 & -16 \\ -28 & 32 & c_{23} \\ -16 & c_{32} & 32 \end{bmatrix} \quad \text{where } c_{23} = c_{32} = 8, -4 \text{ or } -16.$$

if $c_{32} = 8$ $\quad |C| = 4608$

if $c_{32} = -4$ $\quad |C| = -4608$

if $c_{32} = -16$ $\quad |C| = -23040$.

Using Connor's Criterion (1.4.3) it can be concluded that $n(B_2 \cap B_3) = 2$. Also if $n(B_2 \cap B_p) = 4$ then $n(B_1 \cap B_p) = 2$ for the same reason. $(4 \leq p \leq 33)$.

Consider the remaining blocks of the design D as they intersect B_1 and B_2. The major types of blocks are:

Type 1: B_i is of type 1 if it contains 2 varieties from one of B_1 and B_2 and 4 varieties from the other.

Type 2: B_i is of type 2 if it contains 2 varieties from each of B_1 and B_2.

Type 3: B_i is of type 3 if it contains 2 varieties from one of B_1 and B_2 and 3 varieties from the other.

Type 4: B_i is of type 4 if it contains 3 varieties from each of B_1 and B_2.

The above types can be subdivided as follows.

Type 11: B_i contains $\theta_i\theta_j a_k a_\ell$ or $\theta_i\theta_j b_k b_\ell$.

Type 12: B_i contains $\theta_i a_j a_k a_\ell b_m$ or $\theta_i a_j b_k b_\ell b_m$.

Type 21: B_i contains $\theta_i\theta_j$.

Type 22: B_i contains $\theta_i a_j b_k$.

Type 23: B_i contains $a_i a_j b_k b_\ell$.

Type 31: B_i contains $\theta_2\theta_j a_k$ or $\theta_i\theta_j b_k$.

Type 32: B_i contains $\theta_i a_j a_k b_\ell$ or $\theta_i a_j b_k b_\ell$.

Type 33: B_i contains $a_i a_j a_k b_\ell b_m$ or $a_i a_j b_k b_\ell b_m$.

Type 41: B_i contains $\theta_i\theta_j\theta_k$.

Type 42: B_i contains $\theta_i\theta_j a_k b_\ell$.

Type 43: B_i contains $\theta_j a_j a_k b_\ell b_m$.

Type 44: B_i contains $a_i a_j a_k b_\ell b_m b_n$.

Seven equations can now be written down as in the following page. Again x_{ij} is the number of blocks of type ij.

Equation (2.4.1) counts $a_i b_j$ pairs; equation (2.4.2) counts $a_i a_j$ pairs plus $b_i b_j$ pairs; (2.4.3) counts $\theta_i a_j$ pairs plus $\theta_i b_j$ pairs; (2.4.4) counts $\theta_i\theta_j$ pairs; (2.4.5) counts θ varieties; (2.4.6) counts a and b varieties and (2.4.7) counts the blocks B_3 to B_{33}. Note the pairs and elements are counted in the blocks B_3 to B_{33}. This is slightly different from the previous proof.

$$3x_{12} + x_{22} + 4x_{23} + 2x_{32} + 6x_{33} + x_{42} + 4x_{43} + 9x_{44} = 36 \quad (2.4.1)$$

$$x_{11} + 3x_{12} + 2x_{23} + x_{32} + 4x_{33} + 2x_{43} + 6x_{44} = 18 \quad (2.4.2)$$

$$4x_{11} + 4x_{12} + 2x_{22} + 2x_{31} + 3x_{32} + 4x_{42} + 4x_{43} = 90 \quad (2.4.3)$$

$$x_{11} + x_{21} + x_{31} + 3x_{41} + x_{42} = 20 \quad (2.4.4)$$

$$2x_{11} + x_{12} + 2x_{21} + x_{22} + 2x_{31} + x_{32} + 3x_{41} + 2x_{42} + x_{43} = 50 \quad (2.4.5)$$

$$2x_{11} + 4x_{12} + 2x_{22} + 4x_{23} + x_{31} + 3x_{32} + 5x_{33} + 2x_{42} + 4x_{43} + 6x_{44} = 66 \quad (2.4.6)$$

$$x_{11} + x_{12} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} + x_{41} + x_{42} + x_{43} + x_{44} = 31 \quad (2.4.7)$$

Now if (2.4.1), (2.4.2) and (2.4.6) are multiplied by -2, +2, and 1, respectively, and if then they are summed, the result would be

$$(2.4.8) \qquad 4x_{11} + 4x_{12} + x_{31} + x_{32} + x_{33} = 30.$$

But the number of blocks that intersect either B_1 or B_2 in two varieties is at most 11 + 11 = 22. Thus

$$x_{11} + x_{12} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \le 22$$

so $x_{11} + x_{12} + x_{31} + x_{32} + x_{33} \le 22.$

Also $x_{11} + x_{12} = 2$ as $a_4 = 1$ for both B_1 and B_2. So $x_{11} + x_{12} \le 2$

$$\therefore \quad 3(x_{12} + x_{11}) + (x_{12} + x_{11} + x_{31} + x_{32} + x_{33}) \le 28$$

$$4x_{11} + 4x_{12} + x_{31} + x_{32} + x_{33} \le 28.$$

This is a contradiction with (2.4.8) and the following theorem can be stated.

THEOREM 2.4.9. In a balanced incomplete block design with parameters (22, 33, 12, 8, 4) there can be no blocks of class II.

Two Blocks of Class IV Intersecting in Zero Varieties

Since $k + \lambda - r = 0$ in the design (22, 33, 12, 8, 4), Majumdar's Result (1.5.1) can be used to see what structure is forced on the design, if two blocks intersect in zero varieties.

Let two class IV blocks, B_1 and B_2 intersect in zero varieties. Since $a_4 = 1$ for both B_1 and B_2, let B_3 be the block that intersects B_1 in 4 varieties. Thus Majumdar's Theorem states that then $n(B_2 \cap B_3) = 4$. Therefore B_1, B_2 and B_3 can be represented as follows:

$$B_1 = \{\theta_1, \theta_2, \theta_3, \theta_4, a_1, a_2, a_3, a_4\}$$

$$B_2 = \{\phi_1, \phi_2, \phi_3, \phi_4, b_1, b_2, b_3, b_4\}$$

$$B_3 = \{\theta_1, \theta_2, \theta_3, \theta_4, \phi_1, \phi_2, \phi_3, \phi_4\}$$

where the θ, ϕ, a, b are all distinct.

The major type of blocks are:

Type 1: B_i is of type 1 if it contains two varieties from each of B_1 and B_2.

Type 2: B_i is of type 2 if it contains three varieties from each of B_1 and B_2.

Note that there is no type of block which contains two varieties from one of B_1 and B_2 and three from the other. This is specified by Majumdar's Theorem.

Unfortunately each of the above types can be subdivided many times as follows:

Since B_3 consists of θ's and ϕ's, its intersection with B_4 to B_{33} can be uniquely determined. Since B_3 has $a_4 \geq 2$ it must be of class VII, VIII or IX and thus it must intersect every block from B_4 to B_{33} in 1, 2, 3 or 4 varieties. Thus there are no blocks of type 19, 21, 22, 25, 216.

The equations on the following page can now be obtained. Again x_{ij} is the number of blocks of type ij. Equation (2.5.1) counts θ elements; equation (2.5.2) counts ϕ elements; (2.5.3) counts a elements; (2.5.4) counts b elements; (2.5.5) counts $\theta_i\theta_j$ pairs; (2.5.6) counts $\theta_i a_j$ pairs; (2.5.7) counts $\theta_i b_j$ pairs; (2.5.8) counts $\theta_i \phi_j$ pairs; (2.5.9) counts $a_i a_j$ pairs; (2.5.10) counts $a_i b_j$ pairs; (2.5.11) counts $a_i \phi_j$ pairs; (2.5.12) counts $b_i b_j$ pairs; (2.5.13) counts $b_i \phi_j$ pairs; (2.5.14) counts $\phi_i \phi_j$ pairs. Equation (2.5.15) counts all the blocks of type 1 which is equal to $a_2 = 6$. Also equation counts all the blocks of type 2 which is $a_3 = 24$.

Unfortunately these equations have at least 2 solutions if not more. Thus the following theorem is stated:

Type 11: B_i contains $\theta_i\theta_j\phi_k\phi_\ell$

Type 12: B_i contains $\theta_i a_j\phi_k\phi_\ell$

Type 13: B_i contains $a_i a_j\phi_k\phi_\ell$

Type 14: B_i contains $\theta_i\theta_j\phi_k b_\ell$

Type 15: B_i contains $\theta_i a_j\phi_k b_\ell$

Type 16: B_i contains $a_i a_k\phi_k b_\ell$

Type 17: B_i contains $\theta_i\theta_j b_k b_\ell$

Type 18: B_i contains $\theta_i a_j b_k b_\ell$

Type 19: B_i contains $a_i a_j b_k b_\ell$

Type 21: B_i contains $\theta_i\theta_j\theta_k\phi_\ell\phi_m\phi_n$

Type 22: B_i contains $\theta_i\theta_j a_k\phi_\ell\phi_m\phi_n$

Type 23: B_i contains $\theta_i a_j a_k\phi_\ell\phi_m\phi_n$

Type 24: B_i contains $a_i a_j a_k\phi_\ell\phi_m\phi_n$

Type 25: B_i contains $\theta_i\theta_j\theta_k\phi_\ell\phi_m b_n$

Type 26: B_i contains $\theta_i\theta_j a_k\phi_\ell\phi_m b_n$

Type 27: B_i contains $\theta_i a_j a_k\phi_\ell\phi_m b_n$

Type 28: B_i contains $a_i a_j a_k\phi_\ell\phi_m b_n$

Type 29: B_i contains $\theta_i\theta_j\theta_k\phi_\ell b_m b_n$

Type 210: B_i contains $\theta_i\theta_j a_k\phi_\ell b_m b_n$

Type 211: B_i contains $\theta_i a_j a_k\phi_\ell b_m b_n$

Type 212: B_i contains $a_i a_j a_k\phi_\ell b_m b_n$

Type 213: B_i contains $\theta_i\theta_j\theta_k b_\ell b_m b_n$

Type 214: B_i contains $\theta_i\theta_j a_k b_\ell b_m b_n$

Type 215: B_i contains $\theta_i a_j a_k b_\ell b_m b_n$

Type 216: B_i contains $a_i a_j a_k b_\ell b_m b_n$

Equations (1.5.1) to (1.5.16)

$$2x_{11} + x_{12} + 2x_{14} + x_{15} + 2x_{17} + x_{18} + x_{23} + 2x_{26} + x_{27} + 3x_{29} + 2x_{210} + x_{211} + 3x_{213} + 2x_{214} + x_{215} = 40 \quad (.1)$$

$$2x_{11} + 2x_{12} + 2x_{13} + x_{14} + x_{15} + x_{16} + 3x_{23} + 3x_{24} + 2x_{26} + 2x_{27} + 2x_{28} + x_{29} + x_{210} + x_{211} + x_{212} = 40 \quad (.2)$$

$$x_{12} + 2x_{13} + x_{15} + 2x_{16} + x_{18} + 2x_{23} + 3x_{24} + x_{26} + 2x_{27} + 3x_{28} + x_{210} + 2x_{211} + 3x_{212} + x_{214} + 2x_{215} = 44 \quad (.3)$$

$$x_{14} + x_{15} + x_{16} + 2x_{17} + 2x_{18} + x_{26} + x_{27} + x_{28} + 2x_{29} + 2x_{210} + 2x_{211} + 2x_{212} + 3x_{213} + 3x_{214} + 3x_{215} = 44 \quad (.4)$$

$$x_{11} + x_{14} + x_{17} + x_{26} + 3x_{29} + x_{210} + 3x_{213} + x_{214} = 12 \quad (.5)$$

$$x_{12} + x_{15} + x_{18} + 2x_{23} + 2x_{26} + 2x_{27} + 2x_{210} + 2x_{211} + 2x_{214} + 2x_{215} = 48 \quad (.6)$$

$$+2x_{14} + x_{15} + 4x_{17} + 2x_{18} + 2x_{26} + x_{27} + 6x_{29} + 4x_{210} + 2x_{211} + 9x_{213} + 6x_{214} + 3x_{215} = 64 \quad (.7)$$

$$4x_{11} + 2x_{12} + 2x_{14} + x_{15} + 3x_{23} + 4x_{26} + 2x_{27} + 3x_{29} + 2x_{210} + x_{211} = 48 \quad (.8)$$

$$+x_{13} + x_{16} + x_{23} + 3x_{24} + x_{27} + 3x_{28} + x_{211} + 3x_{212} + x_{215} = 18 \quad (.9)$$

$$+x_{15} + 2x_{16} + 2x_{18} + x_{26} + 2x_{27} + 3x_{28} + 2x_{210} + 4x_{211} + 6x_{212} + 3x_{214} + 6x_{215} = 64 \quad (.10)$$

$$+2x_{12} + 4x_{13} + x_{15} + 2x_{16} + 6x_{23} + 9x_{24} + 2x_{26} + 4x_{27} + 6x_{28} + x_{210} + 2x_{211} + 3x_{212} = 64 \quad (.11)$$

$$+x_{17} + x_{18} + x_{29} + x_{210} + x_{211} + x_{212} + 3x_{213} + 3x_{214} + 3x_{215} = 18 \quad (.12)$$

$$+x_{14} + x_{15} + x_{16} + 2x_{26} + 2x_{27} + 2x_{28} + 2x_{29} + 2x_{210} + 2x_{211} + 2x_{212} = 48 \quad (.13)$$

$$x_{11} + x_{12} + x_{13} + 3x_{23} + 3x_{24} + x_{26} + x_{27} + x_{28} = 12 \quad (.14)$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} = 6 \quad (.15)$$

$$+x_{23} + x_{24} + x_{26} + x_{27} + x_{28} + x_{29} + x_{210} + x_{211} + x_{212} + x_{213} + x_{214} + x_{215} = 24 \quad (.16)$$

THEOREM 2.5.17. If two blocks of class IV intersect in zero varieties in a balanced incomplete block design with parameters (22, 33, 12, 8, 4) then a possible structure for the intersection of these two blocks with the remaining blocks of the design is given by:

Solution A: $x_{11} = 2$, $x_{12} = 0$, $x_{13} = 2$, $x_{14} = 0$, $x_{15} = 0$, $x_{16} = 0$, $x_{17} = 2$, $x_{18} = 0$, $x_{19} = 0$, $x_{21} = 0$, $x_{22} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 0$, $x_{26} = 0$, $x_{27} = 8$, $x_{28} = 0$, $x_{29} = 0$, $x_{210} = 8$, $x_{211} = 8$, $x_{212} = 0$, $x_{213} = 0$, $x_{214} = 0$, $x_{216} = 0$, $x_{216} = 0$.

Solution B: $x_{11} = 0$, $x_{12} = 2$, $x_{13} = 1$, $x_{14} = 2$, $x_{15} = 0$, $x_{16} = 0$, $x_{17} = 1$, $x_{18} = 0$, $x_{19} = 0$, $x_{21} = 0$, $x_{22} = 0$, $x_{23} = 1$, $x_{24} = 0$, $x_{25} = 0$, $x_{26} = 0$, $x_{27} = 6$, $x_{28} = 0$, $x_{29} = 1$, $x_{210} = 6$, $x_{211} = 10$, $x_{212} = 0$, $x_{213} = 0$, $x_{214} = 0$, $x_{215} = 0$, $x_{216} = 0$.

Note that in each solution B_3 is a class IX block.

Problems for Further Research

There are a few things that are obviously waiting to be done. A complete set of solutions to the equations of (2.5) should be found. Also all combinations of four variety intersections between class IV, VII, VIII, and IX should be investigated, as in (2.3) or (2.4), to see if any more classes of blocks can be eliminated. Because B_3 was a class IX block in (2.5) the chances of elimination are not too good.

On the other hand, one could try and construct a skelton; that is, take one solution of theorem (2.5.17) and try to put in the actual varieties in B_3 to B_{33} that are common to B_1 or B_2. Then one could try to extend this to a whole design.

However it is not yet clear to me whether it would be more worthwhile trying to prove, by construction, that a (22, 33, 12, 8, 4) design exists or trying to prove that it does not exist.

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