Nearly Orthogonal Latin Squares

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Abstract

A Latin square of order n is an n by n array in which every row and column is a permutation of a set N of n elements. Let $L = [l_{i,j}]$ and $M = [m_{i,j}]$ be two Latin squares of even order n, based on the same N-set. Define the superposition of L onto M to be the n by n array $A = (l_{i,j}, m_{i,j})$. When n is even, L and M are said to be *nearly orthogonal* if the superposition of L onto M has every ordered pair (i, j) appearing exactly once except for $i = j$, when the ordered pair appears 0 times and except for $i - j = n/2 \pmod n$, when the ordered pair appears 2 times. A set of t Latin squares of order $2m$ is called a set of *mutually nearly orthogonal Latin squares* (t MNOLS($2m$)) if the t Latin squares are pairwise nearly orthogonal. We provide two elementary proofs for results that were stated and proved earlier. We also provide some computer results and prove two recursive constructions for MNOLS. Using these results we show that there always exist 3 mutually nearly orthogonal Latin squares of order $2m$, for $2m \geq 358$.

1 Introduction

A *Latin square* of order n is an n by n array in which every row and column is a permutation of a set N of n elements. Unless otherwise stated, we assume $N = \{0, 1, 2, ..., n\}$. Let $L = [l_{i,j}]$ and $M = [m_{i,j}]$ be two Latin squares of order n. Define the superposition of L onto M to be the n by n array $A = (l_{i,j}, m_{i,j})$. Then L and M are said to be *orthogonal* if the superposition of L onto M has every ordered pair (i, j) appearing exactly once. A set of Latin squares in which each pair is orthogonal, is called a set of *mutually orthogonal Latin squares*. Mutually orthogonal Latin squares have been extensively studied (see Dénes and Keedwell [3]). In [6], Raghavarao, Shrikhande and Shrikhande define the concept of two Latin squares being nearly orthogonal. Latin squares L and M are *nearly orthogonal* if the superposition of L onto M has every ordered pair (i, j) appearing exactly once except for $i = j$, when the ordered pair appears 0 times and except for $i - j = n/2 \pmod n$, when the ordered pair appears 2 times. A set of t Latin squares of order $2m$ is called a set of *mutually nearly orthogonal Latin squares*, denoted by t MNOLS($2m$), if the Latin squares

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are pairwise nearly orthogonal. The following two Latin squares of order 4 are nearly orthogonal.

0	1	2	3
1	2	3	0
3	0	1	2
2	3	0	1

1	2	3	0
3	0	1	2
2	3	0	1
0	1	2	3

In [6], Raghavarao, Shrikhande and Shrikhande explained that nearly orthogonal Latin squares are useful for conducting experiments eliminating heterogeneity in two directions and using different interventions each at each level. They went on and proved Theorem 2.1 using eigenvalues for part a) and the theory of group divisible designs for part b). However, the result can easily be proved using elementary counting arguments. Therefore, an elementary proof will be provided in Section 2. In Section 3, we define difference sets for t MNOLS($2m$) and give some computer results concerning these difference sets. In Section 4, we give a direct product construction for NMOLS and a Bose-Shrikhande-Parker-like construction [1] for NMOLS. Then, we combine the computer results and the two constructions to show that 3 MNOLS($2m$) always exist for $2m \geq 358$. In the Section 5 we give some conjectures and problems.

2 Old Results and New Proofs

First we bound the number of Latin squares in a set of MNOLS(2m).

Theorem 2.1 *([6]) Let $m \geq 2$ be a positive integer.*
a)If there exists a set of t MNOLS($2m$), then $t \leq m+1$.
b) If m is even and there exists a set of t MNOLS($2m$), then $t < m+1$.

Proof: a) Suppose there are t MNOLS($2m$) based on the set $N = \{0, 1, \ldots, 2m-1\}$. Let the entry in row i and column j of the superposition of the t MNOLS be $(a^1_{i,j}, a^2_{i,j}, \ldots a^t_{i,j})$, where $a^k_{i,j}$ comes from the k^{th} Latin square. Consider the cells, which have $a^1_{i,j} = 0$ and let $C = \{(i,j)|a^1_{i,j} = 0\}$. In these cells, and for a fixed value of $2 < k \leq t$, the multiset $\{a^k_{i,j}|(i,j) \in C\} = \{1, 2, ..., m-1, m, m, m+1, ..., 2m-1\}$, where the element m occurs twice. Note that m's coming from different Latin squares must occur in different cells, for if they occur in the same cell, there would be two identical elements in the same cell and this is forbidden by definition of nearly orthogonal. Using the fact that the element m occurs twice in positions from C and there are $2m$ cells with $a^1_{i,j} = 0$, we can have at most $2m/2+1 = m+1$ Latin squares in a set MNOLS of order $2m$. Therefore $t \leq m+1$ as required.

b)Suppose that there are $m+1$ MNOLS($2m$) based on the set $N = \{0, 1, \ldots, 2m-1\}$ where m is even. Let the entry in row i and column j of the superposition of the $m+1$ MNOLS be $(a^1_{i,j}, a^2_{i,j}, \ldots a^{m+1}_{i,j})$. Consider the entries from just two of the Latin squares. The $2m$ entries $(0, m), (1, m+1), \ldots, (2m-1, m-1)$ all occur twice in a total of $4m$ distinct cells. We shall call these pairs *special pairs*. If we consider the special pairs between all possible pairs of Latin squares, we see that there are $\binom{m+1}{2} 4m = 2m^3 + 2m^2$ of them. Any particular cell can hold at most $m/2$ special pairs. For if there were $m/2+1$ special pairs in a particular cell, then we would have at least one instance where one special pair comes from Latin squares A and B and the other special pair comes from Latin squares A and C where $B \neq C$. This would imply that the entries from Latin squares B and C are identical at the cell in question, which is not allowed by definition. Hence there must be at least $(2m^3+2m^2)/m/2 = 4m^2+4m$ cells to hold the special pairs. This contradicts the fact that there are only $4m$ cells. ∎

Due to Theorem 2.1, Raghavarao, Shrikhande and Shrikhande [6] define a complete set of MNOLS($2m$) to be a set of t MNOLS($2m$) where $t = m + 1$, if m is odd and where $t = m$, if m is even. Other than for $m = 1$ or 2, complete sets of MNOLS($2m$) seem difficult to construct. In [4], Pasles and Raghavarao proved the following result with the help of a computer.

Theorem 2.2 *([4]) A complete set of 4 MNOLS(6) do not exist.*

The closest approximation to 4 MOLS(6) they got is displayed below without all the commas and brackets.

0134	4310	2503	.	.	.
2415	0253	3140	.	.	.
4251	5142	0325	.	.	.
3520	2035	1452	0341	.	.
1302	1524	4031	.	0413	.
1043	3401	5214	.	.	0532

However, if the cells are filled in a different order, the following approximation to a set of 4 MNOLS(6) can be obtained.

0143	1254	2305	3410	4521	5032
4215	5320	0431	1542	2053	3104
3502	4013	5124	0235	1340	2451
1034	2145	3250	4301	5412	0523
.	.	.	.	.	.
.	.	.	.	.	.

No more cells of the superposed Latin square can be filled in with 4-tuples. Also note that column $i+1$ is obtained from column i by adding 1 mod 6 to every entry. This motivates the following definition by Raghavarao, Shrikhande and Shrikhande [6]. A *(t,2m)-difference set* is a set of $2m$ t-tuples in which the ordered differences modulo $2m$ between elements in two positions form no 0-difference, two m- differences and every other difference appears once. A set of t MNOLS($2m$) can be developed from a $(t, 2m)$ difference set by putting the difference set in the first column (each t-tuple in a different cell) and obtaining each subsequent column from the previous column by adding 1 mod $2m$ to each entry of the cells in the previous column. The difference set { (0,1), (1,3), (3,2), (2,0) } was used to generate the 2 MNOLS(4) stated in the Section 1. Alternatively, the difference set can also be put in the first row and developed along the rows. To save space we will present the difference sets as rows and put the elements from each component of the t-tuple in a different row of our tables. So the 2 MNOLS(4) from the Section 1 could have been developed from the following difference set which would be presented under this scheme as:

0	1	3	2
1	3	2	0

We now state and prove an easy result.

Theorem 2.3 *There exists 2 MNOLS(2m) for all m.*

Proof: It is easy to check that the following is a (2,2m)-difference set.

0	1	2	3	4	...	m-1	m	m+1	...	2m-1
1	3	5	7	9	...	2m-1	0	2	...	2m-2

□

Although there are many ways to obtain $(2, 2m)$-difference sets, it is not easy to provide a systematic way for constructing a $(3, 2m)$-difference set, for any m. In the next section, we give some computer constructions of difference sets.

3 Computer Results

In this section, we try to find some t MNOLS($2m$) for small m. We will be searching for the number of different $(t, 2m)$-difference sets for each m. However, since a difference set can be manipulated to get many different difference sets, we need to define what it means for two difference sets to be isomorphic . Two difference sets are *isomorphic* if one can be obtained from the other with the following operations: 1) rearranging the order of all t-tuples simultaneously, 2) rearranging the rows (assuming the difference set is put in the first column), 3) multiply all elements in all t-tuples of the difference set by a where $\gcd(a, 2m) = 1$, or 4) add a constant to all elements in all t-tuples of the difference set.

Clearly, there is one non-isomorphic set of 2 MNOLS(2). It can be generated from the following $(2, 2)$-difference set: $\{(0, 1), (1, 0)\}$. There is one non-isomorphic $(2, 4)$-difference set which has already been displayed. In [6], they provide a Latin square that could be generated from a $(3, 6)$-difference set. We did a computer search for others and obtained the following lemma.

Lemma 3.1 *There is one non-isomorphic* $(3, 6)$*-difference set which is*

0	*1*	*2*	*3*	*4*	*5*
1	*3*	*5*	*0*	*2*	*4*
3	*0*	*4*	*1*	*5*	*2*

There are no $(4, 6)$*-difference sets.*

Note that the 6 3-tuples come from 2 sets, $\{0, 1, 3\}$ and $\{2, 4, 5\}$. However, we were not able to generalize this idea. Again, in [6], they found an example of 3 MNOLS(8) that could be generated from a $(3, 8)$-difference set.

Lemma 3.2 *There is one non-isomorphic* $(3, 8)$*-difference set which is*

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1	*0*	*5*	*7*	*6*	*3*	*2*	*4*
4	*5*	*3*	*0*	*2*	*7*	*1*	*6*

and there are no $(4, 8)$*-difference sets.*

At this point the computer becomes indispensable.

Lemma 3.3 *There is one non-isomorphic (4,10)-difference set which is*

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	*3*	*7*	*0*	*8*	*4*	*9*	*2*	*6*	*5*
2	*8*	*1*	*6*	*0*	*9*	*7*	*5*	*3*	*4*
7	*5*	*4*	*8*	*9*	*3*	*2*	*6*	*1*	*0*

and there are no (5,10)-difference sets.

Lemma 3.4 *There is more than one non-isomorphic* $(4, 12)$*-difference set. The following is a* $(4, 12)$*-difference set.*

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
1	*0*	*4*	*6*	*9*	*11*	*10*	*2*	*5*	*3*	*8*	*7*
4	*9*	*11*	*10*	*3*	*7*	*0*	*8*	*6*	*2*	*1*	*5*
7	*10*	*5*	*9*	*8*	*1*	*4*	*6*	*2*	*11*	*3*	*0*

and there are no $(5, 12)$*-difference sets.*

At this point we only look for $(3, 2m)$-difference sets for $2m \geq 14$.

Lemma 3.5 *There are* $(3, 2m)$*-difference sets for* $6 \leq 2m \leq 20$*.*

Proof. We need only list the difference sets for $2m = 14, 16, 18$ and 20.

$2m = 14$

0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	4	6	8	11	13	12	2	5	3	9	7	10
2	6	13	10	11	4	12	3	9	7	0	5	1	8

$2m = 16$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	5	7	9	11	13	15	0	2	4	6	8	10	12	14
2	6	0	11	14	1	5	13	12	10	3	8	15	4	9	7

$2m = 18$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	3	5	7	9	11	13	15	17	0	2	4	6	8	10	12	14	16
2	0	7	12	15	1	4	14	11	3	16	8	13	17	5	10	6	9

$2m = 20$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	3	5	7	9	11	13	15	17	19	0	2	4	6	8	10	12	14	16	18
2	0	7	14	17	1	18	8	15	13	19	6	10	16	11	5	4	3	12	9

∎

4 Theoretical Constructions

We now give two recursive construction for NMOLS. First we clarify some notation. If $A = (a_{i,j})$ and if b is an integer then, $A + b = (a_{i,j} + b)$ and $nA = (n * a_{i,j})$. The first construction is of the direct product type.

Theorem 4.1 *Suppose there exists*

1. *s MNOLS(2m),*
2. *s MOLS(n),*
3. *s MOLS(2m)*

Then there exists s MNOLS(2mn).

Proof. Let $B_1, \ldots, B_s$ be the s MOLS(n) based on the set $\{0, 1, ..., n-1\}$, and without loss of generality, we assume that the first row of each of these MOLS is $(0, 1, \ldots, n-1)$. Let $A_1, \ldots, A_s$ be s MNOLS($2m$) based on the set $\{0, 1, \ldots, 2m-1\}$. Let $C_1, \ldots, C_s$ be the s MOLS($2m$) based on the set $S = \{0, 1, \ldots, 2m-1\}$. To construct the desired s MNOLS($2mn$) $L_1, L_2, \ldots, L_s$, we replace each element in the B_i's with a Latin square as follows. For each $1 \leq k \leq s$, replace $B_k(1, j)$ by $nA_k + B_k(1, j) \pmod{2mn}$, where $1 \leq j \leq n-1$ and for each $1 \leq k \leq s$ replace $B_k(i, j)$ by $nC_k + B_k(i, j) \pmod{2mn}$, where $2 \leq i \leq n-1$ and $1 \leq j \leq n-1$.

Since $\bigcup_{i=0}^{n-1} \{0+i, n+i, 2n+i, \ldots, (2m-1)n+i\} = \{0, 1, \ldots, 2mn-1\}$, the L_i's are Latin squares. Consider the pair $(i, mn+i)$ mod $2mn$ coming from two of the Latin squares, say L_p and L_q where $1 \leq p < q \leq s$. Let $i = i_n$ mod n. This pair comes from the subsquares $nA_p + i_n$ and $nA_q + i_n$ that were used to replace $B_p(1, i_n) = i_n$ and $B_q(1, i_n) = i_n$. Since the original MNOLS($2m$) have the pair $((i - i_n)/n, m + (i - i_n)/n)$ twice, $nA_p + i_n$ and $nA_p + i_n$ have the pair $(i, mn+i)$ twice. This uses up all the (i, i) pairs in L_p and L_q so there are no (i, i) pairs in the superposition of L_p and L_q. All other pairs appear once because MOLS and MNOLS were used. Therefore the s Latin squares $L_1, L_2, ..., L_s$ of order $2m$ are s MNOLS($2m$). ∎

The following result is an immediate consequence of Theorem 4.1, where $N(n)$ denotes the cardinality of the largest set of MOLS(n), and $N^*(n)$ denotes the cardinality of the largest set of MNOLS(n).

Corollary 4.2 *Given positive integers m, n, $N^*(2mn) \geq \min\{N(n), N^*(2m), N(2m)\}$.*

Example 4.3 *Using our computer results along with the tables in Colbourn and Dinitz [2], we can get the following two results that we will need later. Since there exists 3 MOLS(4), 3 MNOLS(8) and 3 MOLS(8), then there exists 3 MNOLS(32). Similarly, since there exists 3 MOLS(4), 3 MNOLS(16) and 3 MOLS(16), then there exists 3 MNOLS(64).*

Using the fact [2] that for any $n \geq 11$ there exists 3 MOLS(n), and that there are 3 MNOLS(8) and 3 MOLS(8), Theorem 4.1 implies the following result.

Theorem 4.4 *If $n \geq 11$, then there exists 3 MNOLS($8n$) .*

We would like to prove the existence of 3 MNOLS($2m$) for all 2m larger than some small number, but to do this we need another recursive construction which is similar to the Bose-Shrikhande-Parker construction [1] for MOLS. This will require some definitions which we now do.

Let K be a set of postive integers. A *group divisible design* of order v (K-GDD) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts(groups) whose sizes lie in G and $\mathcal{B}$ is a family of subsets (blocks) of $\mathcal{V}$ which satisfy : 1) If $B \in \mathcal{B}$, then $|B| \in K$, 2) Every pair of distinct elements of $\mathcal{V}$ occurs in 1 block or 1 group but not both, and 3) $|\mathcal{G}| > 1$. If there are a_i groups of size $g_i, i = 1, 2 \ldots s$, then the K-GDD is of *type* $g_1^{a_1}, g_2^{a_2}, \ldots, g_s^{a_s}$.

A t IMOLS(n) is a set of t MOLS(n) with the property that each Latin square is *idempotent*, i.e. the i^{th} row and i^{th} column entry of the Latin square is i.

Theorem 4.5 *Suppose there exists a group divisible design K-GDD of type $g_1^{a_1}, g_2^{a_2}, \ldots, g_s^{a_s}$ Suppose also that for any group of size g_i there are s MNOLS(g_i) and for any block of size $k \in K$ there are s IMOLS(k), then there are s MNOLS($\sum_{i=1}^{s} a_i * g_i = 2h$)*

Proof. We will construct the desired s Latin squares of order $2h$ superposed. Let the rows and columns of this square be indexed by the elements $0, 1, \ldots, 2h$ and for the moment assume the square is empty. The groups of the GDD must all be of even size since there are MNOLS of that size. So it is possible to relabel the $g_i = 2m_i$ elements of each group in the GDD as follows: $\{b_1, b_2, \ldots, b_m, h{+}b_1, h{+}b_2, \ldots, h{+}b_m\} \subset \mathbb{Z}_{2h}$. This relabeling can be done for all groups simultaneously maintaining the property that the groups must partition $\mathcal{V}$. Consider a particular group of size $2m_i$. Map the elements of this group onto the elements of the MNOLS($2m_i$) in such a way that the elements of difference h are mapped onto elements with difference m. Then replace the elements of the MNOLS($2m_i$) with the elements of the groups and superpose the latin squares of order $2m_i$. In that submatrix of the square of order $2h$ picked out by the elements of the group, we will put the s superposed relabeled MNOLS($2m_i$). This will be done for each group. Consider a particular block of size n. Map the elements of this block onto the elements of the IMOLS(n). Then replace the elements of the IMOLS(n) with the elements of the block and superpose the Latin squares of order n. Now delete the main diagonal elements. In that submatrix of the square of order $2h$ picked out by the elements of the block, we will put the s superposed relabeled IMOLS(n) missing the main diagonal. This will be done for each block. We claim the result is a set of s superposed MNOLS($2h$).

Clearly, each element appears exactly once in any row and in any column. Because of the way we relabeled the elements of the groups and the fact that we use the MNOLS on the submatrices picked out by the group elements, any pair of elements with difference h from the distinct Latin squares of order $2h$ appear twice. Since the MNOLS do not have (i, i) pairs and we deleted the main diagonal from the idempotent Latin squares (which contain the (i, i) pairs, there are no (i, i) pairs in the superposed Latin squares of order $2h$. All other pairs in the superposed Latin squares of order $2h$ occur once as all other pairs in the superposed subsquares occur once. □

At this point we need the usual technical lemma.

Lemma 4.6 *All even numbers in the range* $[2^n(11)+6, 2^n(2^n-n+5)]$ *can be written in the form* $2^n m_1 + 2^{n-1} m_2 + \cdots + m_{n-3} 2^4 + s$ *where* $11 \le m_1 \le 2^n - n + 4$, $m_2, m_3, \ldots, m_{n-3}$ *are either 0 or 1,and* $6 \le s \le 20$ *and* s *even.*

Proof. Clearly the even numbers from $2^n(11)+6$ to $2^n(11)+20$ can be expressed in the required form. Then we can add all multiples of 16 from 16 to $2^{n-1}+2^{n-2}+\cdots+2^4 = 2^n-16$ onto the even numbers $2^n(11)+6$ to $2^n(11)+20$ to show that all even numbers from $2^n(11)+22$ to $2^n(11)+2^n+4$ can be expressed in the desired form. We can then add positive multiples of 2^n from 2^n to $2^n(2^n-n-7)$ to the numbers $2^n(11)+6$ to $2^n(11)+2^n+4$ to show that all even numbers from $2^n(11)+2^n+6$ to $2^n(11)+2^n+4+2^n(2^n-n-7) = 2^n(2^n-n+5)$ can be expressed in the required form. Hence all even numbers from $2^n(11)+6$ to $2^n(2^n-n+5)$ can be expressed in the required form. ∎

For a fixed value of n, the n^{th} *range* is defined to be the set of all possible numbers of the form $2^n m_1 + 2^{n-1} m_2 + \cdots + m_{n-3} 2^4 + s$ where $11 \le m_1 \le 2^n - n + 4$, $m_2, m_3, \ldots, m_{n-3}$ are either 0 or 1,and $6 \le s \le 20$ and s even.

Corollary 4.7 *All even numbers equal to or larger than 358 can be written in the form* $2^n m_1 + 2^{n-1} m_2 + ... + 2^4 + s$ *where* $11 \le m_1 \le 2^n - n + 4$, $m_2, m_3, \ldots, 4$ *are either 0 or 1,and* $6 \le s \le 20$, s *even and* $n \ge 5$.

Proof. If we examine the ranges in Lemma 4.6 we see that the largest number in the n^{th} range (i.e. $2^n(2^n-n+5)-16$) is larger the smallest number in the $(n+1)^{th}$ range (i.e. $2^{n+1}(11)+6$) for $n \ge 5$. Since the smallest number in the n^{th} range, when $n=5$ is 358, we get the result. ∎

We are now ready for our main Theorem.

Theorem 4.8 *If* $2m \ge 358$, *then there exist 3 MNOLS(*$2m$*).*

Proof. By Corollary 4.7, we know that any number equal to or larger than 358 can be written in the form $2^n m_1 + 2^{n-1} m_2 + ... + 2^4 m_{n-3} + s$ where $11 \le m_1 \le 2^n - n + 4$, $m_2, m_3, \ldots, m_{n-3}$ are either 0 or 1, $6 \le s \le 20$, s even and $n \ge 5$. As 2^n is a prime power, we know that there exist $2^n - 1$ MOLS (2^n). Also in [2], we know how to convert these $2^n - 1$ MNOLS(2^n) into a 2^n+1-GDD of type $(2^n)^{2^n+1}$. We note that this GDD has the property that every block intersects every group in exactly one element.

First, we shorten one of the groups to size s, i.e. delete $2^n - s$ elements from the group and delete these elements whenever they appear in a block. Second, for $2 \le i \le n-3$, if $m_i = 0$, delete one of the unaltered groups and its elements entirely from the GDD. Third, for $1 \le i \le n-3$, if $m_i = 1$, then shorten one of the unaltered groups to 2^{n-i+1} elements. At this point there are $2^n - n + 4$ unaltered groups. Delete $2^n - n + 4 - m_1$ of the unaltered groups entirely from the GDD. This leaves exactly m_1 unaltered groups which we leave alone.

This process produces a K-GDD with m_1 groups of size 2^n, at most $2^n - n + 3$ groups have a size that is a power of two larger than 8 and one group of size s. Since there exist 3 MNOLS(32), 3 MNOLS(64) (by Example 4.3) and 3 MNOLS($2m$) for $2m = 0 \bmod 8$ (by Theorem 4.4), there exist 3 MNOLS for the powers of 2 that are sizes of the groups. From our computer results in Section 3, we also know that there 3 MNOLS($2m$), for $6 \le 2m \le 20$.

Since $m_1 \geq 11$, we know that each block intersects at least 11 groups and hence the blocks have sizes greater than or equal to 11. So we know that there are 3 IMOLS for all block sizes (see [2]). Now we apply Theorem 4.5 to get the result. ∎

5 Conclusions and Conjectures

We have shown that if $2m \geq 358$, then there exist 3 MNOLS($2m$). It is interesting to compare this particular existence question to the one for 3 MOLS. It is possible to reduce 358 with some work. We conjecture the following:

Conjecture 5.1 *There exist 3 MNOLS($2m$) for $2m \geq 6$.*

Clearly one could study t MNOLS($2m$) for $t \geq 4$ and all the other problems that are mentioned in the MOLS section of Colbourn and Dinitz [2].

Based on our computer results in section 3, we conjecture the following.

Conjecture 5.2 *The maximum t for which a $(t, 2m)$-difference set exists is* $\lceil m/4 \rceil + 1$

Another interesting question is the existence of complete set of MNOLS. Except for $2m = 2, 4$, there are none known and we know that there is no complete set of MNOLS(6). It would be interesting to investigate whether a complete set of MNOLS exists for any value of $2m$ other than $2, 4$.

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