A note on the completion of partial latin squares

Nicholas J. Cavenagh, Diane Donovan
Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Queensland 4072
Australia

G.H.J. van Rees
Department of Computer Science
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

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In this note we introduce a new test by which the completion of a partial latin square to a latin square may be "forced".

1 Introduction

The problem of completing partial latin squares to latin squares of the same order has been studied for many years. For instance, in 1960 Evans **[9]** conjectured that every partial latin square of order n containing at most $n-1$ filled cells is completable to a latin square of order n. This conjecture was shown to be true by Lindner **[12]** and Smetaniuk **[13]**. Recently, Bryant and Rodger **[6]** established necessary and sufficient conditions for completing an arbitary 2 by n latin rectangle to an n by n symmetric latin square. Colbourn [8] demonstrated that the problem of determining whether an

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arbitrary partial latin square has a completion is NP-complete. In this paper, we will focus on the study of the unique completion of partial latin squares. Such ideas are important for the determination of critical sets in latin squares **[2], [5]**.

2 Partial latin squares and arrays of alternatives

A *partial latin square* is an n by n array in which each entry of $N = \{1,\ldots,n\}$ occurs at most once in each row and at most once in each column. The *size* of the partial latin square is taken to be the number of non-empty cells. For convenience, we may represent a partial latin square as a subset of the set of ordered triples $N \times N \times N$. That is, the ordered triple (i,j,k) represents the occurrence of the entry value k in the cell representing the intersection of row i with column j. For a partial latin square P of order n, and each $a \in N$, $\mathcal{R}_P^a$ and $\mathcal{C}_P^a$ are defined to be the sets:

$$\begin{aligned} \mathcal{R}_P^a &= \{x \in N \mid (a,j,x) \in P \text{ for some } j \in N\} \\ \mathcal{C}_P^a &= \{x \in N \mid (i,a,x) \in P \text{ for some } i \in N\} \end{aligned}$$

If each entry of N occurs exactly once in each row and once in each column then the array is said to be a *latin square.* A partial latin square P of order n may be *completed* to a latin square L of order n, if $P \subseteq L$. If P is a subset of a unique latin square L of order n, then P has *unique completion* to L, or equivalently P is *uniquely completable* (UC) to L.

If P is a partial latin square, we define a *conjugate* (or *parastrophy*) of P to be one of the partial latin squares given by:

$$\begin{array}{l} \{(j,k,i) \mid (i,j,k) \in P\}, \{(k,i,j) \mid (i,j,k) \in P\}, \\ \{(i,k,j) \mid (i,j,k) \in P\}, \{(k,j,i) \mid (i,j,k) \in P\}, \\ \{(j,i,k) \mid (i,j,k) \in P\}, \{(i,j,k) \mid (i,j,k) \in P\}. \end{array}$$

Thus in a conjugate of a partial latin square, rows may become columns, columns may become entries, entries may become rows and vice versa. Taking a conjugate preserves the combinatorial structure of a partial latin square, in particular the property of having a unique completion.

There has been much discussion about the process of completing partial latin squares and, in the literature, some ambiguity over what terms are used to describe the different methods by which a partial latin square may be completed to a latin square. With this note we attempt to provide a more unified approach.

Given any partial latin square P of order n, we may define an *array of alternatives* for P, denoted by A_P, as follows:

1. A_P is an $n \times n$ array with each cell containing either a subset of $N = \{1, \ldots, n\}$ or an asterisk $*$.
2. If $(i, j, k) \in P$ for some entry k, the cell (i, j) in A_P contains an asterisk $*$.
3. If for all $k \in N$, $(i, j, k) \notin P$, (i.e. the cell (i, j) is empty in P), then the cell (i, j) in A_P contains the set $N \setminus (\mathcal{R}_P^i \cup \mathcal{C}_P^j)$ (i.e. the set of elements of N that appear in neither row i nor column j of P). It is possible that this set is empty, in which case P does not have a completion to a latin square.

Like partial latin squares, we may also consider an array of alternatives as a set of ordered triples. If $M \subseteq N$ or $M = *$ occurs in cell (i, j) of A_P, then we say that $(i, j, M) \in A_P$. Furthermore if $(i, j, M) \in A_P$, we say that $(i, j)_{A_P} = M$.

A partial latin square $Q5$ of order 5, with corresponding array of alternatives A_{Q5} is presented below. Note the partial latin square has size 7.

1	2			
2				
			1	
		5	3	
				3

Partial latin square $Q5$

$*$	$*$	3, 4	4, 5	4, 5
$*$	1, 3, 4, 5	1, 3, 4	4, 5	1, 4, 5
3, 4, 5	3, 4, 5	2, 3, 4	$*$	2, 4, 5
4	1, 4	$*$	$*$	1, 2, 4
4, 5	1, 4, 5	1, 2, 4	2, 4, 5	$*$

Array of alternatives A_{Q5}

For a given partial latin square P of order n and its array of alternatives A_P, we say that entry value k is *redundant* in $(i, j)_{A_P}$ if for all latin squares L of order n such that $P \subseteq L$, we have $(i, j, k) \notin L$. The literature contains a number of papers which study this 'forcing out process', see for instance **[1, 4, 11]**. The majority of these papers show that a specific partial latin square P is UC by sequentially removing redundant entry values from the array of alternatives. The paper **[1]** is of interest as it presents many examples of partial latin squares of order 5 and 6 for which the identification of redundant entry values is not immediately obvious. In an endeavour to understand this process of UC, authors have categorised redundant entry values into three broad classes, 'strongly forced out', 'semi-strongly forced out' and 'weakly forced out'. The interesting examples presented in **[1]** fall into the category of redundant entry values which are weakly forced out. With this paper we seek to present a unified approach to the process of removing redundant entry values out of an array of alternatives. The ideas presented here will be of interest to researchers studying critical sets in latin squares, defining sets in block designs and defining sets and forcing sets in graphs.

3 Forcing out

Let P be a partial latin square of order n and A_P its array of alternatives. Our discussion will focus on key subsets T of the array of alternatives A_P. For each such specified subset T and a given entry $x \in N$, we define $ET_x = \{(i,j) \mid (i,j,M) \in T \text{ and } x \in M\}$. Further we let $t_x(T)$ represent the maximum size of a partial latin square which is a subset of $\{(i,j,x) \mid (i,j) \in ET_x\}$. In other words, $t_x(T)$ is the maximum number of times we can place entry x in cells from ET_x without repetition in a row or column.

We say that k is *forced out* of $(i,j)_{A_P}$ if in the array of alternatives $A_{P\cup\{(i,j,k)\}}$ there exists a subset T of $A_{P\cup\{(i,j,k)\}}$ such that

$$\sum_{x\in N} t_x(T) < |T|. \tag{1}$$

If such a k exists then we define $\mathcal{A} - (i,j,k)$ to be the *reduced array of alternatives* as follows:

$$\mathcal{A} - (i,j,k) = \{(r,c,M) \in A_P \setminus \{(i,j,(i,j)_{\mathcal{A}})\} \cup \{(i,j,(i,j)_{\mathcal{A}} \setminus \{k\})\}.$$

For example, consider the partial latin square $Q5$. The array of alternatives for the partial latin square $Q5 \cup \{(4,2,4)\}$ is displayed below. Note that for the subset $T = \{(4,1,\emptyset)\}$

$$\sum_{x\in N} t_x(T) = 0 < 1 = |T|.$$

Thus Condition (1) is satisfied, implying entry 4 is forced out of cell $(4,2)$ of A_{Q5}.

$*$	$*$	$\{3,4\}$	$\{4,5\}$	$\{4,5\}$
$*$	$\{1,3,5\}$	$\{1,3,4\}$	$\{4,5\}$	$\{1,4,5\}$
$\{3,4,5\}$	$\{3,5\}$	$\{2,3,4\}$	$*$	$\{2,4,5\}$
$\emptyset$	$*$	$*$	$*$	$\{1,2\}$
$\{4,5\}$	$\{1,5\}$	$\{1,2,4\}$	$\{2,4,5\}$	$*$

$A_{Q5\cup\{(4,2,4)\}}$

This specific example illustrates the concept of k being *strongly forced out* of cell (i,j) of the array of alternatives. In general, if

- for some $(i',j') \neq (i,j)$, but $i' = i$ or $j' = j$, $T = \{(i',j',\emptyset)\}$ is a subset of the array of alternatives for $P' \cup \{(i,j,k)\}$,

(where P' is some conjugate of P) then the entry value k is *strongly forced out* of cell (i,j) of the array of alternatives. That is, the subset T is a single cell and Condition (1) corresponds to

$$\sum_{x\in N} t_x(T) = 0 < 1 = |T|. \tag{2}$$

This restriction is equivalent to stating that there exists i', j', k such that $(i', j', \{k\}) \in A_{P'}$, where P' is some conjugate of P.

For a more complicated example, we may consider the partial latin square $P6$ given below. (We omit set brackets in A_{P6} for presentation purposes.) Note this example appeared earlier in [11].

6	1	2	3		
1	2	3	4		
2	3	4			1
3	4				
					3
				3	

Partial latin square $P6$

$*$	$*$	$*$	$*$	$4,5$	$4,5$
$*$	$*$	$*$	$*$	$6,5$	$6,5$
$*$	$*$	$*$	$6,5$	$6,5$	$*$
$*$	$*$	$6,1,5$	$6,1,2,5$	$6,1,2,5$	$6,2,4,5$
$4,5$	$6,5$	$6,1,5$	$6,1,2,5$	$N \setminus \{3\}$	$*$
$4,5$	$6,5$	$6,1,5$	$6,1,2,5$	$*$	$6,2,4,5$

Array of alternatives A_{P6}

$*$	$*$	$*$	$*$	$*$	4
$*$	$*$	$*$	$*$	6	$6,5$
$*$	$*$	$*$	$6,5$	6	$*$
$*$	$*$	$6,1,5$	$6,1,2,5$	$6,1,2,5$	$6,2,4,5$
$4,5$	$6,5$	$6,1,5$	$6,1,2,5$	$N \setminus \{3\}$	$*$
$4,5$	$6,5$	$6,1,5$	$6,1,2,5$	$*$	$6,2,4,5$

Array of alternatives $A_{P6\cup\{(1,5,5)\}}$

The setset $T = \{(2,5,\{6\}),(3,5,\{6\})\}$ of the array of alternatives for the partial latin square $P6 \cup \{(1,5,5)\}$ satisfies Condition (1) as follows,

$$\sum_{x\in N} t_x(T) = 1 < |T| = 2.$$

Consequently entry 5 is forced out of cell $(1,5)$ of A_{P6}. This example illustrates the concept of k being *semi-strongly forced out* of cell (i,j) of the array of alternatives. In general, if

- for some subset $\{i_1, \dots i_r\}$ of N with $i \notin \{i_1, \dots i_r\}$,

  $$T = \{(i_1, j, M_1), \dots, (i_r, j, M_r)\}$$

  where $k \in \cup_{1\le z\le r} M_z$ and $|\cup_{1\le z\le r} M_z| < r$ is a subset of the array of alternatives for $P' \cup \{(i,j,k)\}$, (where P' is some conjugate of P) then the entry value k is *semi-strongly forced out* of cell (i, j) of the array of alternatives. In this case the subset T is a $1 \times r$ subarray and Condition (1) corresponds

$$\sum_{x \in N} t_x(T) = s < r = |T|. \tag{3}$$

This restriction is equivalent to stating that given a partial latin square P' which is some conjugate of P and the associated array of alternatives $A_{P'}$, if for some $r \geq 1$ and $i_1, \ldots, i_r$ (all $\neq i$) there exists $k \in (i,j)_{A_{P'}}$, with $k \in (i_1, j)_{A_{P'}} \cup \cdots \cup (i_r, j)_{A_{P'}}$ and $|(i_1, j)_{A_{P'}} \cup \cdots \cup (i_r, j)_{A_{P'}}| = r$, then k is forced out of $(i,j)_{A_{P'}}$. So in the above example we have $\{(2, 5, \{6, 5\}), (3, 5, \{6, 5\})\}$ of A_{P6} with $|\{(2, 5, \{6, 5\}), (3, 5, \{6, 5\})\}| = 2$ and $|\{5, 6\}| = 2$.

In the literature if k is forced out of cell (i, j) of the array of alternatives, but is not semi-strongly forced out, then it has been termed *weakly forced out*. However as the next two examples illustrate that under certain circumstances Condition (1) can still be applied to identify entries which are clearly 'forced out'.

Consider the following partial latin square $Q12$ of order 12, together with its array of alternatives $\mathcal{A}_{Q12}$. There is no entry in $\mathcal{A}_{Q12}$ which is semi-strongly forced out.

		4	8	6		9	10	11	12	5	7
		8	10	7	6	3	9	12	11		5
5				1	8	6	7	10	9	11	12
7	5			8	10		6	9	3	12	11
2	8	5				11	12	7	6	9	10
8	10	7	5			12	11	6		3	9
11	12	10	9		5			1	8	7	6
12	11	9	2	5	7			8	10	6	
	6	11	12	10	9	7	5			8	2
6	7	12	11	9	4	5				10	8
9	1	6	7	11	12	10	8		5		
10	9		6	12	11	8	4	5	7		

Partial latin square $Q12$

1,3	2,3	*	*	*	1,2,3	*	*	*	*	*	*
1,4	2,4	*	*	*	*	*	*	*	*	1,2,4	*
*	2,3,4	2,3	3,4	*	*	*	*	*	*	*	*
*	*	1,2	1,4	*	*	1,2,4	*	*	*	*	*
*	*	*	1,3,4	3,4	1,3	*	*	*	*	*	*
*	*	*	*	2,4	1,2	*	*	*	1,2,4	*	*
*	*	*	*	2,3,4	*	2,4	2,3	*	*	*	*
*	*	*	*	*	*	1,4	1,3	*	*	*	1,3,4
1,3,4	*	*	*	*	*	*	*	3,4	1,4	*	*
*	*	*	*	*	*	*	1,2,3	2,3	1,2	*	*
*	*	*	*	*	*	*	*	2,3,4	*	2,4	3,4
*	*	1,2,3	*	*	*	*	*	*	*	1,2	1,3

Array of alternatives $\mathcal{A}_{Q12}$

However, take $T = \{(1,1,\{1,3\}),(1,2,\{3\}),(2,1,\{1,4\}),(2,2,\{4\})\}$, a subset of the array of alternatives for the partial latin square $Q12 \cup \{(3,2,2)\}$. Now Condition (1) corresponds to

$$\sum_{x \in N} t_x(T) = 3 < |T| = 4.$$

Hence entry 2 is forced out of $(3,2)_{A_{Q12}}$. In a similar fashion we can show that entry 3 is also forced out of $(3,2)_{A_{Q12}}$. Thus if L is latin square of order 12, such that $Q12 \subseteq L$, then $(3,2,4) \in L$.

Finally we apply the above theory to a more complicated example.

Take $P5$ to be the partial latin square shown below and let A_{P5} be its associated array of alternatives.

				5
	1	4		
3		5		
4		2		
			2	4

Partial latin square $P5$

1,2	2,3,4	1,3	1,3,4	*
2,5	*	*	3,5	2,3
*	2,4	*	1,4	1,2
*	3,5	*	1,3,5	1,3
1,5	3,5	1,3	*	*

Array of alternatives A_{P5}

We begin by noting that no entry value can be strongly forced out of A_{P5}. However, the subset $T_1 = \{(4,2,\{5\}),(5,2,\{5\})\}$ of the array of alternatives for $P \cup \{(1,2,3)\}$ can be used to semi-strongly force entry value 3 out of cell $(1,2)$ of the A_{P5}. This gives the reduced array of alternatives $A_{P5} - (1,2,3)$. At this point it is not possible to semi-strongly force any entry out of $A_{P5} - (1,2,3)$. Despite this it is possible to show that entry value 1 can forced out of cell $(1,4)$. Let T_2 be the subset of array of alternatives for $P5 \cup \{(1,4,1)\}$ displayed below. For this subset we note that

$$\sum_{x \in N} t_x(T_2) = 5 < |T_2| = 6.$$

Hence entry 1 is forced out of cell $(1,4)$ of A_{P5} producing the reduced array of alternatives $(A_{P5} - (1,2,3)) - (1,4,1)$. which for brevity we will denote by RA_{P5}.

	2, 4		4	1, 2
	3, 5		3, 5	1, 3

Subarray T_2 of $A_{P5 \cup \{(1,4,1)\}}$

1, 2	2, 4	1, 3	3, 4	
2, 5			3, 5	2, 3
	2, 4		1, 4	1, 2
	3, 5		1, 3, 5	1, 3
1, 5	3, 5	1, 3		

RA_{P5}

Next consider the subset T_3 of the array of alternatives for the $P \cup \{(1,4,3)\}$.

		1		
			5	
	3, 5		1, 5	1, 3
	3, 5	1, 3		

Subarray T_3 of $A_{P \cup \{(1,4,3)\}}$

Observe that $t_1(T_3) = t_3(T_3) = t_5(T_3) = 2$ and $t_2(T_3) = t_4(T_3) = 0$. We see that for this subarray Condition (1) corresponds to

$$\sum_{x \in N} t_x(T_3) = 6 < \alpha = 7.$$

Hence entry value 3 is force out of cell $(1,4)$ of the array of alternatives giving a reduced array of alternatives $((A_{P5} - (1,4,1)) - (1,2,3)) - (1,4,3)$. Thus if L is a latin square of order 5, such that $P5 \subseteq L$, We must have $(1,4,4) \in L$.

4 Strong, semi-strong and weak UC sets

In the literature the above arguments have been extended to classify partial latin squares as strongly UC, semi-strongly UC and weakly UC. We now show how Condition (1) can be used in this process.

Let P be a partial latin square of order n, with UC to a latin square L of order n, and let P_i, $0 \leq i \leq s$, where $s = |L| - |P| - 1$, be a sequence of partial latin squares such that

- $P_0 = P$ and $P_s = L$, and
- $P_{i+1} = P_i \cup \{(\alpha, \beta, \gamma)\}$, for some $(\alpha, \beta, \gamma) \in L$.

If $0 \leq i < s$ implies that Condition (1) can be used to force entry values out of the array of alternatives for P_i (or some conjugate of P_i) to produce the array of alternatives for P_{i+1}, then we say that the unique completion of P to L has been *forced*.

If for each $i \in \{0, \ldots, s\}$ the entry values have been strongly forced out (i.e. Condition (2) is applied) then we say that P has *strong UC* to L. If for each $i \in \{0, \ldots, s\}$ entry values have been semi-strongly forced out (i.e. Condition (3) is applied) then we say that P has *semi-strong UC* to L. If there exists an $i \in \{0, \ldots, s\}$ such that one of the entries have been weakly forced out then we say that P has *weak UC* to L.

We conclude with the following open problem.

Open Problem 1 *Does there exist a UC partial latin square P, such that in the array of alternatives A_P, no entry can be initially* forced out *(as defined in this paper)?*

So far we have not discovered a partial latin square with this property. If there exists no such partial latin square, then this paper implies a forcing algorithm for weakly UC partial latin squares.

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