The Existence of Non-Resolvable Steiner Triple Systems

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Abstract

We consider two well-known constructions for Steiner triple systems and show that by using the right ingredients, the resultant systems are non-resolvable. The first construction we consider is the standard $v \to 2v+1$ construction. The other construction we consider is the Wilson construction.

Keywords:
Steiner triple systems, resolvability, combinatorial designs

1 Introduction

A *Steiner triple system* (STS) is a set system $(X, \mathcal{B})$, where X is a finite set of *points* and $\mathcal{B}$ is a set of 3-element subsets of X called *triples* or *blocks*, such that each pair of distinct points of X appear in exactly one triple in $\mathcal{B}$. The *order* of $(X, \mathcal{B})$ is the cardinality of X. We denote a Steiner triple system of order v by STS(v). A Steiner triple system $(X, \mathcal{B})$ is *resolvable* if the triples of $\mathcal{B}$ can be partitioned into *parallel classes*, where each parallel class is a partition of X. If such a partitioning is not possible, then the system is said to be *non-resolvable*.

In 1847, The Reverend Thomas P. Kirkman [3] showed that a STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$. In 1971, Ray-Chaudhuri and Wilson [5] showed that a resolvable STS(v) exists if and only if $v \equiv 3 \pmod 6$. Since then, resolvable Steiner triple systems have been extensively studied. For a good up-to-date summary of results, see [1] or [2].

To our knowledge, it has not been shown that a non-resolvable STS(v) exists for all $v \equiv 3 \pmod 6$, $v > 9$. Yet it is almost universally accepted that such non-resolvable STS(v)s exist. One reason is that, for a given v, the number of non-isomorphic STS(v)s is at least $v^{v^2+\left(\frac{1}{6}+o(1)\right)}$ [2] and most of these are probably non-resolvable.

In this paper, we will consider two well-known constructions which can be used to construct non-resolvable Steiner triple system. One of the constructions is recursive while the other is direct.

A related, but much more difficult question is that of the existence of Steiner triple systems of order v where $v > 9$ and $v \equiv 3 \pmod 6$ without a single parallel class. It is conjectured in [2] (Conjecture 19.4) that such systems exists for all $v > 9$ and $v \equiv 3 \pmod 6$. We do not attempt to answer this question.

2 A Recursive Construction of Non-Resolvable Steiner Triple Systems

In this section, we focus our attention our attention on constructing non-resolvable STS(v)s for $v \equiv 3 \pmod{12}$. We begin by stating a well-known construction of a STS($2v+1$) using a STS(v), for $v \equiv 1 \pmod 6$. The interested reader is to refer to [2] for a more detail description of this construction.

Construction 2.1 *: Suppose* $v \equiv 1$ *(mod* 6*) and let* $(X, \mathcal{B})$ *be a STS(*v*), where* $X = \{1, 2, ..., v\}$ *. Let* $n = (2v+1) - |X| = v+1$*. Clearly,* n *is an even number. Let* $Y = \{v+1, v+2, ..., 2v+1\}$ *and construct the complete graph* K_n *whose vertices are labelled using* Y*. Consider a 1-factorization* $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_v\}$ *of* K_n *and for each factor* $\mathcal{F}_i$*,* $1 \leq i \leq v$*, add the element* i *to each pair in* $\mathcal{F}_i$ *and call the resulting set of triples containing* i*,* $\mathcal{G}_i$*. The set system* $(X \cup Y, \mathcal{B} \cup (\cup_{i=1}^{v} \mathcal{G}_i)$ *is clearly STS*$(2v+1)$*.*

Unfortunately, the STS$(2v+1)$ constructed may be very difficult to work with, in terms of showing non-resolvability. Hence, we need to specify this construction so that the STS contains a Pasch configuration. A *Pasch configuration* is a set of four triples of the form $\{\{a, b, c\}, \{a, x, y\}, \{b, x, z\}, \{c, y, z\}\}$. Then we show that by replacing a Pasch configuration in the STS$(2v+1)$ with a different Pasch configuration, that is, $\{\{x, y, z\}, \{x, a, b\}, \{y, a, c\}, \{z, b, c\}\}$ we obtain a non-resolvable STS$(2v+1)$.

Theorem 2.2 *There exists an STS(*$2v+1$*) that is not resolvable, for* $v \equiv 1$ *(mod* 6*).*

Proof. First we must specify Construction 2.1 to ensure that there is a Pasch configuration in the STS. Let $\{x, y, z\}$ be a block in $\mathcal{B}$ and let a, b, c be elements in Y. In the construction, we must ensure that elements x, y, z are put with three distinct 1-factors containing the pairs ab, ac, bc, respectively. It is easy to check that this process is always possible. Then the set of blocks $\{\{x, y, z\}, \{x, a, b\}, \{y, a, c\}, \{z, b, c\}\}$ is a Pasch configuration. We replace these four triples by $\{\{a, b, c\}, \{a, x, y\}, \{b, x, z\}, \{c, y, z\}\}$. Clearly, the resulting system is still a Steiner triple system. We now proceed to show that it is not resolvable. Assume it is resolvable and let P_1 be the parallel class that contains the triple $\{a, x, y\}$. Since none of the triples $\{\{a, b, c\}, \{b, x, z\}, \{c, y, z\}\}$ may appear in P_1, the triples that do appear in P_1 contain either zero or two elements from Y. Therefore P_1 contains an odd number of elements from Y, implying that $|Y|$ is odd which is a contradiction. Hence, the design is not resolvable. □

The following result is an immediate consequence of Theorem 2.2.

Corollary 2.3 *There exists a non-resolvable STS(*v*) for* $v \equiv 3$ *(mod* 12*),* $v > 3$*.*

Unfortunately, we were not able to recursively construct a non-resolvable STS(v) for $v \equiv 9 \pmod{12}$. An attempt was made to use the $v \to 2v+7$ construction outlined in [2], but we were unsuccessful.

3 A Direct Construction of Non-Resolvable Steiner Triple Systems

In this section, we will show that the construction of Wilson [6] can give a non-resolvable STS(v) whenever $v \equiv 3 \pmod 6$, $v > 9$. Thoughout this section, the terminologies used will be consistent with those found in [4]. We begin by re-stating the Wilson construction for Steiner triple systems.

Construction 3.1 (**The Wilson Construction**) *Let* $v \equiv 1, 3 \pmod 6$*.* $X = \{0, 1, 2, ..., v-3, \infty_1, \infty_2\}$*. We construct three different types of triples:*
Type 1 *: The set of triples* $\mathcal{T}^* = \{\{x, y, z\} : x + y + z \equiv 0 \pmod{v-2}$*), where* x, y, z *are distinct elements in* $\mathbb{Z}^*_{v-2} = \{1, 2, ..., v-3\}$*.*
Type 2 *: On the pairs from* $\mathbb{Z}^*_{v-2}$ *not appearing in the triples of Type 1, construct a 1-factorization of these pairs consisting of the three 1-factors denoted by* $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$*. Define the Type 2 triples to be following set of triples:* $\mathcal{T}_0 = \{\{0, x, y\} : \{x, y\} \in \mathcal{F}_0\}$*,* $\mathcal{T}_1 = \{\{\infty_1, x, y\} : \{x, y\} \in \mathcal{F}_1\}$*,* $\mathcal{T}_2 = \{\{\infty_2, x, y\} : \{x, y\} \in \mathcal{F}_2\}$*.*
Type 3 *: The triple* $T = \{0, \infty_1, \infty_2\}$*.*
The set system $(\mathbb{Z}^*_{v-2} \cup \{0, \infty_1, \infty_2\}, \mathcal{T}^* \cup \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{T\})$ *is an STS(v).*

The key to showing that the Wilson Construction can give non-resolvable systems is to find a "good" 1-factorization of the pairs that do not appear in the Type 1 triples. These edges form a graph called the *deficiency graph*. By choosing the 1-factors of the deficiency graph in an appropriate manner, we can force certain Type 2 triples to appear with other Type 2 triples of our choosing. An important fact about the pairs not appearing in the Type 1 triples is that they are exactly all the pairs of the form $\{\{x, -x\}, \{x, -2x\} : x \in \mathbb{Z}^*_{v-2}\}$. It is clear that there are exactly $\frac{3(v-3)}{2}$ edges in the deficiency graph, $\frac{v-3}{2}$ edges in the deficiency graph of the form $\{x, -x\}$ and $v-3$ edges of the form $\{x, -2x\}$. It should be noted that the edges of the form $\{x, -x\}$ have sum $\equiv 0 \pmod{v-2}$. In addition, the edges of the form $\{x, -2x\}$ have sum $\equiv -x \pmod{v-2}$, and hence $\bigcup_{x \in \mathbb{Z}^*_{v-2}} \{(-2x + x) \pmod{v-2}\} = \mathbb{Z}^*_{v-2}$. We now give several examples using The Wilson Construction for $v = 15, 21$.

Example 3.2 *Suppose* $v = 15$*. The pairs from* $\mathbb{Z}^*_{13}$ *not appearing in the Type 1 triples form the deficiency graph shown in Figure 1. A particular 1-factorization of the deficiency graph is illustrated in Figure 1. Notice that all the edges of the form* $\{x, -x\}$ *appear in one of the 1-factors. In the remaining two 1-factors, an edge* $\{x, -2x\}$ *appears in the 1-factor if and only if the edge* $\{-x, 2x\}$ *does. It can be verified that the* $STS(15)$ *obtained is non-resolvable.*

Example 3.3 *Suppose* $v = 21$*. The pairs from* $\mathbb{Z}^*_{19}$ *not appearing in the Type 1 triples form the deficiency graph shown in Figure 2. A particular 1-factorization of the deficiency graph is illustrated in Figure 2. It can be verified that the* $STS(21)$ *obtained is non-resolvable.*

Wilson [6] proved that the deficiency graph may have one or more connected components, each of which is 3-regular. In addition, the components must be either a wheel or a bi-wheel. A *wheel* is a graph consisting of a cycle of even length and opposite vertices are joined by an edge. A *bi-wheel* (or a *prism graph*) is a graph consisting of two cycles (called inner and outer cycles) of the

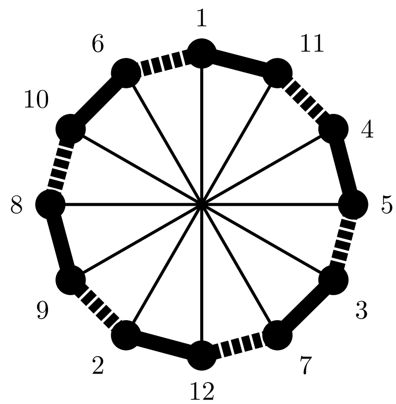

Figure 1: The Deficiency Graph modulo 13 with a Particular 1-Factorization

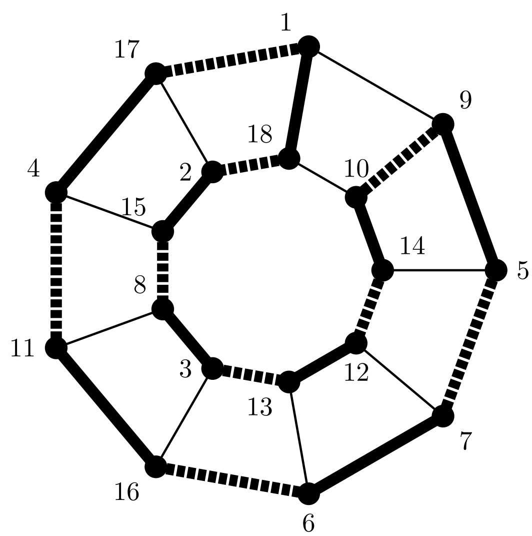

Figure 2: The Deficiency Graph modulo 19 with a Particular 1-Factorization

same length where each vertex of the outer cycle is adjacent to a vertex of the inner cycle. In a bi-wheel, an edge between a vertex of the inner cycle and a vertex of the outer cycle is called a *spoke*. It should be noted at this time that the definition of a wheel should not be confused with a *wheel graph* from graph theory. We now state a simple result that is useful in showing that a system is non-resolvable.

Lemma 3.4 *Suppose the Steiner triple system* $(X, \mathcal{B})$ *of the Wilson Construction is resolvable. Let* $\{0, x, y\}$, $\{\infty_1, z, w\}$, $\{\infty_2, u, t\}$ *be part of a parallel class that are Type 2 triples. Then* $x + y + z + w + u + t \equiv 0 \pmod{v-2}$.

Proof. The remaining triples that fill up this parallel class must be Type 1 triples. The sum of the elements of each of these Type 1 triples is 0 (mod $v-2$). Now, the elements 1, 2, ..., $v-3$ must appear exactly once in the parallel class and their sum is $\frac{(v-3)(v-2)}{2}$, which is a multiple of $v-2$, as $v-3$ is even. Hence it must be that $x + y + z + w + u + t \equiv 0 \pmod{v-2}$. □

The following result is a consequence of Lemma 3.4.

Lemma 3.5 *Suppose the Steiner triple system* $(X, \mathcal{B})$ *of the Wilson Construction is resolvable. Let* $\{0, x, -x\}$, $\{\infty_1, z, w\}$, $\{\infty_2, u, t\}$ *be part of a parallel class. Then* $z + w + u + t \equiv 0 \pmod{v-2}$. *In addition, it must be that* $z = y$, $w = -2y$, $u = -y$ *and* $t = 2y$ *for some* $y \in \mathbb{Z}_{v-2}^*$, *unless* $z = -w$ *and* $u = -t$.

Proof. The first part is obvious from Lemma 3.4. If $z = -w$, then $u = -t$ and there is nothing to prove. So let $z \neq -w$ and without loss of generality let $z = y$ and $w = -2y$. Then $u + t = y$ which implies that one of u or t is $2y$ and the other $-y$. □

We now begin with a simple case of when the Wilson Construction will give a non-resolvable design.

Theorem 3.6 *If the deficiency graph of the Wilson Construction consists of a single wheel and* $4|(v-3)$, *then the 1-factors in the Wilson Construction can be chosen so that the Steiner triple system is non-resolvable.*

Proof. Since the deficiency graph is a single wheel, take for the 1-factor $\mathcal{F}_0$ all the edges of the form $\{x, -x\}$. Take as the remaining two 1-factors, $\mathcal{F}_1$ and $\mathcal{F}_2$, the alternating edges of the cycle of length $v-3$. Since $4|(v-3)$, then the number of edges in each 1-factor is even. Thus, if $\{x, -2x\} \in \mathcal{F}_1$ then $\{-x, 2x\} \in \mathcal{F}_1$. Then Lemma 3.5 implies that the STS constructed with this particular 1-factorization of the deficiency graph is not resolvable. □

It should be noted that if the 1-factors are selected as described in Theorem 3.6, the STS(v) constructed will not only be non-resolvable, but a parallel class in the system must contain the triple $\{0, \infty_1, \infty_2\}$. This immediately implies that the constructed system consists of at most one parallel class. This parallel class must consists of the triple $\{0, \infty_1, \infty_2\}$ along with $v/3 - 1$ triples from the Type 1 triples.

The following result characterizes the Pasch configuration of the Steiner triple system constructed using the 1-factorization given in Theorem 3.6 and whose deficiency graph consists of a single wheel and $4|(v-3)$.

Theorem 3.7 *Consider the STS(v) constructed using the Wilson Construction whose deficiency graph consists of a single wheel where* $4|(v-3)$ *and the 1-factorization used in the construction is the one used in Theorem 3.6. Then there are exactly* $v-3$ *Pasch configurations. There are* $\frac{v-3}{2}$ *Pasch configurations consisting of the four Type 2 triples* $\{\{0, x, -x\}, \{0, 2x, -2x\}, \{\infty_1, x, -2x\}, \{\infty_1, -x, 2x\}\}$. *The remaining* $\frac{v-3}{2}$ *Pasch configurations consists of the four Type 2 triples* $\{\{0, x, -x\}, \{0, 2x, -2x\}, \{\infty_2, x, -2x\}, \{\infty_2, -x, 2x\}\}$.

Proof. It is clear that the Pasch configurations consisting of only Type 2 triples are accounted for in the theorem. We now need to show that these are the only ones. Consider a candidate configuration which contains the Type 3 triple $\{0, \infty_1, \infty_2\}$. The other candidate triples must be Type 2 triples and be of the form $\{0, x, y\}$, $\{\infty_1, x, z\}$ and $\{\infty_2, y, z\}$. It is clear that this is not possible since there are no triangles in the deficiency graph. Now, consider a candidate configuration consisting entirely of Type 1 triples. Let the triples be $\{\{x, y, z\}, \{x, u, t\}, \{y, u, w\}, \{z, t, w\}\}$ where x, y, z, u, t, w are distinct and from $\mathbb{Z}_{v-2}^*$. Since they are Type 1 triples, they satisfy the following equations:

$$
\begin{aligned}
x + y + z &\equiv 0 \pmod{v-2} \\
x + u + t &\equiv 0 \pmod{v-2} \\
y + u + w &\equiv 0 \pmod{v-2} \\
z + t + w &\equiv 0 \pmod{v-2}
\end{aligned}
$$

Subtracting the the 2nd, 3rd, 4th equations from the 1st, we get $-2u - 2t - 2w \equiv 0 \pmod{v-2}$ and hence $2(u+t+w) \equiv 0 \pmod{v-2}$. Since $v-2$ is odd, we have $u+t+w \equiv 0 \pmod{v-2}$. If we compare this to the third equation we see that $t \equiv y \pmod{v-2}$. This is a contradiction since we assumed that t and y are distinct. Hence no Pasch configuration exists entirely of only Type 1 triples. The only remaining cases to check are when there are i Type 2 triples and $4-i$ Type 1 triples in a candidate Pasch configuration, where $1 \le i \le 3$. If i=3 then this would imply that the deficiency graph contains a triangle which is still a contradiction. If $i = 1$ then 0 occurs only once which is a contradiction. If i=2, then the configuration contains $\{0, x, -x\}$, $\{0, y, -y\}$ where $x \neq \pm y$. However, the two remaining Type 1 triples force x to equal $\pm y$. □

We now consider the case where the deficiency graph is a single bi-wheel.

Theorem 3.8 *If the deficiency graph of the Wilson construction consists of a single bi-wheel, then the 1-factorization of the deficiency graph can be chosen so that Steiner triple system is non-resolvable.*

Proof. There are two possible cases to consider.
Case 1: Consider the case where the inner cycle (and hence outer cycle) have even length in the deficiency graph. In this case take as one 1-factor $\mathcal{F}_0$ the spokes of the deficiency graph. These edges consists of all the edges of the form $\{x, -x\}$. Take as the remaining two 1-factors $\mathcal{F}_1$, $\mathcal{F}_2$ alternating edges in the inner and outer cycles. Clearly, $\{x, -2x\}$ and $\{-x, 2x\}$ are in the same 1-factor, for all x. Then Lemma 3.5 implies that this gives a non-resolvable Steiner triple system.
Case 2 : Consider the case where the inner and outer cycles have odd length in the deficiency graph. In this case, let y be any vertex in the graph. Take as one 1-factor $\mathcal{F}_0$ the edges $\{\{x, -x\} : x \in \mathbb{Z}^*_{v-2}\} \cup \{\{y, -2y\}, \{-y, 2y\}\} \setminus \{\{y, -y\}, \{2y, -2y\}\}$. Take as another 1-factor, $\mathcal{F}_1$ the edge $\{y, -y\}$ and the the alternating edges in the inner and outer cycles starting at $\{2y, -4y\}$ and $\{-2y, 4y\}$. Then the remaining one 1-factor, $\mathcal{F}_2$, contains the edge $\{2y, -2y\}$ and the alternating edges in the inner and outer cycles starting at $\{4y, -8y\}$ and $\{-4y, 8y\}$. If $v-2 > 6$, then there must be a parallel class containing $\{0, x, -x\}$ and $\{\infty_1, y, -2y\}$. But the pair $\{-y, 2y\}$ also occurs with ∞_1, so Lemma 3.5 implies that this STS is not resolvable. □

We now consider the case where the deficiency graph is a single wheel in which $4 \not| v-3$.

Theorem 3.9 *If the deficiency graph of the Wilson construction consists of a single wheel and $4 \not|(v-3)$, then the 1-factors in the Wilson Construction can be chosen so that the Steiner triple system is non-resolvable.*

Proof. Let y be any vertex in the graph. Take as one 1-factor $\mathcal{F}_0$ the edges $\{\{x, -x\} : x \in Z^*_{v-2}\} \cup \{\{y, -2y\}, \{-y, 2y\}\} \setminus \{\{y, -y\}, \{2y, -2y\}\}$. Take as the 1-factor, $\mathcal{F}_1$, the edge $\{y, -2y\}$ and the $\frac{v-5}{2}$ alternating rim edges starting at $\{-2y, 4y\}$ and the $\frac{v-5}{2}$ alternating rim edges starting at $\{-y/2, y/4\}$. Let $\mathcal{F}_0$ be the 1-factor containing the remaining edges of the deficiency graph. If $v-3 > 3$ then there is a triple $\{0, x, -x\}$ that is in a parallel class with $\{\infty_1, x, -2x\}$. But the pair $\{-x, 2x\}$ occurs with ∞_1 also, so Lemma 3.5 implies this STS is non-resolvable. □

Note that for $v = 9$, the STS constructed by the Wilson Construction is resolvable. Incidentally, it turns out that the first occurrence of the deficiency graph consisting of a single wheel and $4 \not|(v-3)$ is during the construction of an STS(81).

At this point, we are ready to handle the cases where there are more than one connected component in the deficiency graph. Before we do this, we need the following lemma.

Lemma 3.10 *For all $v > 9$ and $v \equiv 3 \pmod 6$, there exists a cycle of the form $x \to -2x \to 4x \to \ldots \to x$ of length greater than six in the deficiency graph with $v-2$ vertices.*

Proof. We can check values of $v \leq 70$ where $v \equiv 3 \pmod 6$ individually by hand. For all $v > 70$ and $v \equiv 3 \pmod 6$, since $(-2)^6 = 64 < v - 2$ and hence the cycle $1 \to -2 \to 4 \to ... \to 1$ has length greater than six. □

Lemma 3.10 allows us to avoid the case where all the components in the deficiency graph is made up entirely of wheels or bi-wheels on six or fewer points. Some of these two types of components, when our constructions are applied to them, do not allow the 1-factorizations to have one of the 1-factors with a majority of the edges of the form $\{x, -x\}$.

Theorem 3.11 *If the deficiency graph of the Wilson Construction consists of more than one component, then the 1-factors in the Wilson Construction can be so chosen that the Steiner triple system is non-resolvable.*

Proof. The cases where $v \leq 70$, are easy to check. So assume $v > 70$. For each component C_i with v_{C_i} vertices of the deficiency graph, find a 1-factorization of three 1-factors, using the techniques in Theorems 3.6, 3.8, 3.9. One of the 1-factors (which we'll call F_i^*) will contain at least $\frac{v_C}{2} - 2$ edges of the form $\{x, -x\}$. Construct a 1-factorization of the deficiency graph by taking as one of the 1-factors $G_1 = \cup_i F_i^*$. Since $v > 70$, Lemma 3.10 implies a cycle of length greater than six, which implies that G_1 will contain a majority of the edges of the form $\{x, -x\}$. The remaining two 1-factors G_2, G_3 can be chosen in any number of ways. Note that edges $\{x, -2x\}$ and $\{-x, 2x\}$ occur in the same component and hence are in the same one factor. Since G_1 contains the majority of the edges of the form $\{x, -x\}$, there will be a triple $\{0, y, -y\}$ that appears with a triple $\{\infty_1, z, -2z\}$. Since the pair $\{-z, 2z\}$ also occurs with ∞_1 (in the triple $\{\infty_1, -z, 2z\}$) , Lemma 3.5 implies that the this STS is non-resolvable. □

In the following example, we show how the deficiency graph for an STS(33) can be 1-factorized to give a non-resolvable design.

Example 3.12 *Suppose* $v = 33$. *The deficiency graph is given by Figure 3. Using the technique in Theorem 3.9, construct a 1-factorization for each of the three components. Each component will have a factor which consists of 3 edges of the form* $\{x, -x\}$. *Take these factors from each component and combine them to form one 1-factor of the entire graph. This 1-factor of the deficiency graph contains 9 edges of the form* $\{x, -x\}$ *and 6 edges of the form* $\{x, -2x\}$. *It is easy to verify that this gives a non-resolvable STS*(33).

4 Conclusion

We can summarize the results from Section 3 with the following result.

Corollary 4.1 *For all* $v \equiv 3 \pmod 6$, $v > 9$, *there exists a non-resolvable STS(v).*

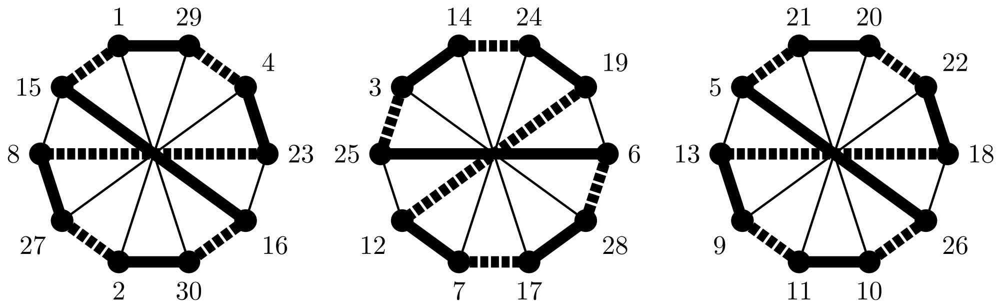

Figure 3: The Deficiency Graph modulo 31 with an appropriate 1-factorization

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