

A Note on Critical Sets

J. A. Bate and G. H. J. van Rees*

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

bate@cs.umanitoba.ca vanrees@cs.umanitoba.ca

Abstract

First, a counterexample to a published theorem on critical sets in Latin squares is given. Second, an example is given showing that a published theorem on critical sets in F-squares can not be strengthened.

1 Counterexample

A *Latin square*, L , of order n is an $n \times n$ array with symbols chosen from a set N of size n , such that each element of N occurs precisely once in each row and column. So L may be thought of as a set of n^2 triples $(i, j; k)$, where the cell (i, j) of L contains the symbol k . Latin squares and their properties have been much studied. A *partial Latin square*, P , of order n is an $n \times n$ array with entries from a set N of size n , such that each element of N occurs at most once in each row and each column. Again P may be thought of as a set of p triples which correspond to the p filled cells in P where $0 \leq p \leq n^2$. One property of interest is the size of the smallest partial Latin square that uniquely defines a Latin square of a specified order. More precisely, a

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	2		4
3		1	
4			1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Figure 1: critical set C and Latin square L

subset of the entries of a Latin square, L , of order n is called a *critical set*, C , of L if L is the only Latin square of order n containing all the entries from C and any proper subset of C is contained in at least two distinct Latin squares. A long list of papers studying critical sets can be found in Gower [5]. Cooper, Donovan and Seberry [2] define a *strongly* critical set of a Latin square, L , with symbols from the set N , to be a critical set with the property that there exists a set $\{P_1, P_2, \dots, P_f\}$ of partial Latin squares of order n with $f = n^2 - |C|$ such that:

1. $C = P_1 \subset P_2 \subset \dots \subset P_f = L$
2. for any i , $2 \leq i \leq f$, where $P_i = P_{i-1} \cup (r_{i-1}, s_{i-1}, t_{i-1})$, the set $P_{i-1} \cup (r_{i-1}, s_{i-1}, t'_{i-1})$ is not a partial Latin square for any $t' \in N - t$.

Fitina, Seberry and Chaudhry [3] have tried to study these critical sets more deeply. In order to do this they define the nest of a critical set. Let B be a subset of a critical set C in a Latin square, L , of order n . The *nest* of a B , $N(B)$, is defined to be the union of $C - B$ and the largest set that can be uniquely filled from $C - B$. They define a set of triples, A to be *uniquely filled* from a set $X \subseteq L$, if A is a subset of every Latin square of order n which contains X . They claim to have proved the following theorem:

Theorem 1.1 *If x and y are any two triples in C , a critical set of a Latin square L , then $N(\{x, y\}) = N(\{x\}) \cap N(\{y\})$.*

Certainly, if x and y are triples in C , then $N(\{x, y\}) \subseteq N(\{x\}) \cap N(\{y\})$ is true as they proved. Unfortunately $N(\{x, y\}) \supseteq N(\{x\}) \cap N(\{y\})$ is false. Consider the critical set C in Figure 1. L is the only Latin square that contains C . Let x be $(4,4;1)$ and let y be $(3,3;1)$, then Figure 2 shows $N(\{x\})$, $N(\{y\})$, and $N(\{x, y\})$.

Clearly, $N(\{x, y\}) \neq N(\{x\}) \cap N(\{y\})$ and their theorem is not true. This theorem was an important part of their paper. The rest of the paper must be read with care and some scepticism.

1	2	3	4
2		4	
3	4	1	2
4		2	

1	2	3	4
2			3
3			2
4	3	2	1

1	2	3	4
2			
3			
4			

Figure 2: Nest of $\{x\}$ Nest of $\{y\}$ Nest of $\{x, y\}$

2 An example

Another type of square that has been studied is the Frequency square or F-square. An *F-square* of type $F = F(n; \alpha_0, \alpha_1, \dots, \alpha_{v-1})$ is an $n \times n$ array with symbols chosen from the set $N = \{0, 1, \dots, v-1\}$ such that each element i occurs α_i times in each row and in each column where $n = \alpha_0 + \alpha_1 + \dots + \alpha_{v-1}$. As in Latin squares, an F-square can be thought of a set of triples $(i, j; k)$ where cell (i, j) contains element or symbol k . Again as in Latin squares, researchers are interested in the size of the smallest set of entries that uniquely defines an F-square of a specified type.

Define a *critical set* of an F-square to be a non-empty subset S of an F-square, FF of type $F = F(n; \alpha_0, \alpha_1, \dots, \alpha_{v-1})$ if FF is the only F-square of type F which has element k in position (i, j) for each $(i, j; k) \in S$ and every proper subset of S is contained in at least two F-squares of type F . One way to find critical sets in larger F-squares is by using products of Latin squares and products of critical sets. Let L be a Latin square of order n and let J_2 be the two by two square of all ones. Then $L \times J_2$ is the F-square of order $2n$ consisting of 4 copies of L and $C \times J_2$ is a partial square of $L \times J_2$ as is shown in the following:

L	L	C	C
L	L	C	C

Fitina and Seberry [4] proved the following theorem.

Theorem 2.1 *If C is a strongly critical set of a Latin square L , then $C \times J_2$ is a critical set of F-square- $(n; 2, 2, \dots, 2)$, $L \times J_2$.*

It would be nice if the word strongly could be deleted from the theorem. But this is impossible as the next example shows. In Figure 3 C is a critical set of a Latin square L . This example is taken from

				5
	1	4		
3				
4		2		
			2	

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

Figure 3: critical set C and Latin square L

2	4	1	3	5	1	2	3	4	5
5	1	4	3	2	2	1	4	5	3
3	2	5	4	1	3	4	5	1	2
4	5	2	1	3	4	5	2	3	1
1	3	5	2	4	5	3	1	2	4
1	2	3	4	5	2	3	1	4	5
2	1	4	5	3	5	1	4	3	2
3	4	1	5	2	3	2	5	1	4
4	3	2	1	1	4	5	2	5	3
5	5	3	2	4	1	4	3	2	1

Figure 4: The F-square M

Adams and Khodkar [1]. Now, $C \times J_2$ is embedded in the F-square $L \times J_2$ but also in the F-square M shown in Figure 4.

References

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