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Some non-isomorphic $(4t + 4, 8t + 6, 4t + 3, 2t + 2, 2t + 1)$-BIBD's

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Abstract

Kocay, W. and G.H.J. van Rees, Some non-isomorphic $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's, Discrete Mathematics 92 (1991) 159–172.

In this paper the automorphism groups of parts of the $(4t+3, 2t+1, t)$-BIBD's are studied in order to show that these designs can be used to construct a great many non-isomorphic $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's if certain conditions are met. In particular, 92,436,200 non-isomorphic (16, 30, 15, 8, 7)-BIBD's 10,869,917,004,894,316 non-isomorphic (20, 38, 19, 10, 9)-BIBD's, 403,880,965,784,264,348 non-isomorphic (24, 46, 23, 12, 11)-BIBD's, 9,820,329,245,540,352,000,000 non-isomorphic (28, 54, 27, 14, 13)-BIBD's, and 38,029,083,844,041,728,836,746,735,484 non-isomorphic (32, 62, 31, 16, 15)-BIBD's are constructed.

1. Introduction

A *balanced incomplete block design* (BIBD) is a pair (V, B), where V is a v-set of elements called *varieties* and B is a collection of b k-subsets of V, called *blocks* such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. The parameters of a BIBD are denoted (v, b, r, k, λ). Trivial necessary conditions for the existence of a (v, b, r, k, λ)-BIBD are:

$$vr = bk, \qquad r(k-1) = \lambda(v-1).$$

Two designs (V_1, B_1) and (V_2, B_2) are *isomorphic* if there exists a bijection $\theta : V_1 \to V_2$ such that $B_1^\theta = B_2$, where B_1^θ indicates that the mapping θ is to be applied to each element of each block of B_1. Isomorphism of designs is denoted

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by $(V_1, B_1) \cong (V_2, B_2)$. An isomorphism θ of a design $D = (V, B)$ with itself is called an *automorphism*. The set of all automorphisms of D forms a group, the *automorphism group* of D, denoted Aut(D). It is a subgroup of the *symmetric* group Sym(V), of all permutations of V.

In recent years, there has been much interest shown in the number of mutually non-isomorphic designs on a given parameter set. Much of this interest has been stimulated by the tables of Mathon and Rosa [9]. Clearly the parameters $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$ obey the necessary conditions. Also, such designs are known to exist for many values of t, see Hall [6]. Much work has been done to increase the number of known designs on these parameters for small t, see [4], [5], or [10], for instance. In this paper, we increase the bounds on the number of such BIBD's known for $t = 3, 4, 5, 6$, and 7. Before we do this however, we need to introduce some terminology. Any missing terms can be found in Hall [6].

A *symmetric* BIBD, denoted SBIBD, is a BIBD for which $v = b$, and consequently, $r = k$. A BIBD is *resolvable* if there exists a partition R of its set of blocks B into subsets called *parallel classes* each of which partitions the set V; R is called a *resolution* of the design. A $t-(v, k, \lambda)$ design is a BIBD (V, B), in which each t-subset of V is contained in exactly λ blocks. So a (v, b, r, k, λ)-BIBD is a $2-(v, k, \lambda)$ design. The word *design* will be used in its generic sense, denoting any combinatorial configuration of blocks.

The *incidence matrix* A of a (v, b, r, k, λ)-BIBD is the $b \times v$ matrix whose entry in the ith row and jth column is 1 if the jth element of V is in the ith block of B, and 0 otherwise. The elements of V are usually numbered $1, 2, 3, \ldots, v$, sometimes with an ∞ element. The blocks are always numbered $1, 2, \ldots, b$. The *dual* of a BIBD $D = (V, B)$, with incidence matrix A, is the BIBD with incidence matrix A', the transpose of A. The *complement* of D, denoted $\bar{D}$, is obtained by replacing each block of D by its complement with respect to V. The incidence matrix of $\bar{D}$ is $\bar{A}$.

A *Hadamard* matrix is a $(1, -1)$-matrix whose row vectors are orthogonal. If the first row and column consists of all 1's, then the matrix is said to be *normalized*. It is always possible to normalize a Hadamard matrix (in several ways), since the orthogonality is preserved by multiplying a row or column by -1, or by permuting rows or columns.

A *cyclic* design is a design D whose automorphism group contains a cyclic subgroup $\Gamma \leqslant \mathrm{Aut}(D)$ which acts transitively on the blocks (that is, any block can be mapped to any other). Such a design D can be generated from any one of its blocks, say B, by applying the elements of Γ to it. B is said to be a *base block* for D.

We can represent a design $D = (V, B)$ by its *Levi graph*, which is a bipartite graph with vertex set $V \cup B$ in which each vertex $i \in V$ is adjacent to all blocks of B containing i. We can then use the graph isomorphism program of Kocay [8] to compute Aut(D), and a *certificate* for D. A certificate is a character string which uniquely identifies the design D up to isomorphism, so that isomorphic designs will have equal certificates, but a non-isomorphic design will have a different certificate from D. We have used the certificates produced by the program of [8] to identify isomorphism classes of designs.

With these definitions, we state and prove in Section 2 theorems which show that $(4t+3, 2t+1, t)$-SBIBD's can be used to construct many mutually non-isomorphic $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's. In Section 3, we discuss the case of $t=3$ and its history. In Section 4, we do the same for $t=4$. Finally, in Section 5, we show how to construct a great many non-isomorphic BIBD's, for $t=5$, 6, and 7. It is straightforward to check whether the constructions discussed can be used for particular values of t.

2. General results

In order to produce non-isomorphic $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's from $(4t+3, 2t+1, t)$-BIBD's, we need to investigate the structure of these latter designs thoroughly. First we give the standard construction for these designs in the following two lemmas.

Lemma 1. *The existence of a* $(4t+3, 2t+1, t)$*-design is equivalent to the existence of a resolvable* $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$*-BIBD.*

Proof. Let N be the incidence matrix of a $(4t+3, 2t+1, t)$-SBIBD. Then $\bar{N}$ is the incidence matrix of the complement of the SBIBD. It is easily verified that $\begin{pmatrix} N & \underline{1} \\ \bar{N} & \underline{0} \end{pmatrix}$, where $\underline{1}$ and $\underline{0}$ are column vectors of all 1's and 0's, respectively, is the incidence matrix of a resolvable BIBD. The element corresponding to the $\underline{1}$ column is often denoted by ∞.

Conversely, the incidence matrix of a resolvable BIBD with the above parameters can be put in the above form by a suitable permutation of its rows and columns, where N is a square matrix of order $4t+3$ having $2t+1$ elements equal to 1 in each row and column. Using a result of Hall [6], the resolvable design is affine-resolvable, that is, the blocks of N must pairwise intersect in

$$\frac{(2t+2)^2}{4t+4} - 1 = t$$

elements. So the dual of N, and hence N itself, are $(4t+3, 2t+1, t)$-SBIBD's. □

Lemma 2. *If* D_1 *and* D_2 *are any* $(4t+3, 2t+1, t)$-SBIBD's *with incidence matrices* N_1 *and* N_2*, respectively, then* $\begin{pmatrix} N_1 & \underline{1} \\ N_2 & \underline{0} \end{pmatrix}$ *is the incidence matrix of a* $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$*-BIBD.*

Proof. Check the parameters. □

Any $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD that is constructed from two $(4t+3, 2t+1, t)$-SBIBD's D_1 and D_2 in such a fashion will be denoted by $D_1\bar{D}_2$. The next lemma is known but not much exploited. It can be found in Beth, Jungnickel, and Lenz [1]. We give a self-contained proof.

Lemma 3. *D is a resolvable $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD if and only if D is a 3-$(4t+4, 2t+2, t)$ design.*

Proof. Consider the blocks of a 3-design D. Let one of them be B. Let x_i be the number of other blocks that intersect B in i varieties. Then the following equations can be found by counting blocks, elements, pairs of elements, or triples of elements in two ways.

$$\sum x_i = 8t+5,$$

$$\sum ix_i = 2(2t+1)(2t+2),$$

$$\sum i^2x_i = (2t+1)(2t+2)^2,$$

$$\sum i^3x_i = 2(t+1)^2(2t+1)(2t+2).$$

Now

$$\sum (i-t)(i-(t+2))x_i = \sum i^2x_i - (2t+2)\sum ix_i + t(t+2)\sum x_i$$
$$= t^2-6t-4. \qquad (1)$$

Also $\sum i(i-t)(i-(t+2))x_i = -2(2t+1)(2t+2)$ so that $-(t+1)x_{t+1} \leqslant -2(2t+1)(2t+2)$, which implies that $x_{t+1} \geqslant 8t+4$ and $x_i = 0$ for $i \neq 0$, t, or $t+2$. From (1), $t(t+2)x_0 - x_{t+1} = t^2-6t-4$. So for x_0 to be an integer, the only solution is $x_0 = 1$, $x_{t+1} = 8t+4$, and $x_i = 0$, for $i \neq 0$ or $t+1$. Since this is true for any block of the 3-design, the 3-design is a resolvable $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD.

Now consider the set of equations relating to block intersections of a 2-design. They are the same as before, except that the last equation need not hold. However the 2-design is resolvable, so $x_0 \geqslant 1$. Then there is only one solution to the equations, which happens to be the same as before. Now let us count the triples of elements, and let one of the elements of the triple be ∞, as in Lemma 1. Then clearly the triple occurs as often as the λ of the 2-design, i.e., t times. □

Corollary 4. *The number of non-isomorphic 3-$(4t+4, 2t+2, t)$ designs is equal to the number of non-isomorphic, resolvable $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's.*

Now Bhat [2] has suggested that many non-isomorphic designs $D_i\bar{D}_j$, may be obtained by applying permutations to one of the two constituent designs. This we plan to do, but first we need some terminology and notation.

Let $D=(V, B)$ be a BIBD. If $\sigma \in \mathrm{Sym}(V)$, then D^σ represents the design obtained from D under the action of the permutation σ. We will construct designs of the form $D_i\bar{D}_j^\sigma$.

The set of blocks containing element i is denoted $D(i)$. It is called the *derivative* of D at i. The set of blocks not containing i is $D(\bar{i})$, the *anti-derivative* of D at i. Also, $\bar{D}(i)$ (resp., $\bar{D}(\bar{i})$) will be the set of blocks of $\bar{D}$ containing (resp., not containing) the element i. If a new element, usually denoted ∞, is adjoined to each block of a set D of blocks, then we define this to be the *extended* design D^*. Two $(4t+3, 2t+1, t)$-SBIBD's, D_1 and D_2, are said to be *compatible* if there do not exist any i and j such that $D_1(\bar{i})^* \cong \bar{D}_2(j)$, that is, an extended anti-derivative in D_1 is isomorphic to a derivative in the complement of D_2. It is necessary to use the extended anti-derivate, because each block of $\bar{D}_2(j)$ contains the point j. (Note: $D_1(\bar{i})^* \cong \bar{D}_2(j)$ implies $D_2(\bar{j})^* \cong \bar{D}_1(i)$, since $\overline{D(\bar{i})} = \bar{D}(i)$.) If D_1 and D_2 are not compatible, they are said to be *incompatible*. Two $(4t+3, 2t+1, t)$-SBIBD's are said to be *agreeable* if there does not exist i and j such that $D_1(i) \cong D_2(j)$, i.e., no derivative of D_1 is a derivative of D_2.

The following lemma is stated for its intrinsic interest.

Lemma 5. *Any* $D_i(\bar{j})^*$ *is isomorphic to some* $\overline{D_k(\bar{l})}$, *where* D_i *and* D_k *are* $(4t+3, 2t+1, t)$*-SBIBD's. (The complement of every anti-derivative is also an anti-derivative.)*

Proof. Any $(4t+3, 2t+1, t)$-SBIBD can be embedded in a normalized Hadamard matrix of dimension $4t+4$ as follows. Let N_i be the incidence matrix of D_i wherein all the 0's have been replaced by -1's. Then if $\underline{1}$ is a column vector of $4t+3$ 1's,

$$H = \begin{pmatrix} 1 & \underline{1}^{\mathrm{T}} \\ \underline{1} & N_i \end{pmatrix}$$

is a normalized Hadamard matrix of dimension $4t+4$. By permuting rows, we can create the situation where the first column is all 1's and the second column consists of $2t+2$ entries equal to 1 followed by $2t+2$ entries equal to -1. Consider the last $2t+2$ rows of this matrix. If the -1's are treated as 0's, these rows are the incidence matrix of some extended anti-derivative $D_i(\bar{j})^*$, and every anti-derivative of D_i can be so obtained. Multiply these rows by -1 and interchange the first two columns of H in order to renormalize it. This complements the blocks of $D_i(\bar{j})^*$, giving $\overline{D_i(\bar{j})}$. But since H has been renormalized, these rows are now the incidence matrix of some $D_k(\bar{l})$. Note that D_i and D_k are not necessarily isomorphic, as the Hadamard matrix is now normalized on a different column. Also, notice that $\overline{D_i(\bar{j})} = \bar{D}_i(j)$. $\square$

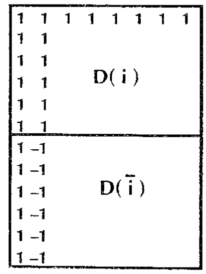

Fig. 1.

Lemma 6. *Let D_i and D_j be compatible $(4t+3, 2t+1, t)$-SBIBD's with the same variety set V, and let $\sigma, \tau \in \mathrm{Sym}(V.)$ If the designs $D_i\bar{D}_j^{\sigma}$ and $D_i\bar{D}_j^{\sigma\tau}$ are isomorphic, with isomorphism θ, then $\theta \in \mathrm{Aut}(D_i)$ and $\theta(\infty) = \infty$.*

Proof. Let $\theta : D_i\bar{D}_j^{\sigma} \rightarrow D_i\bar{D}_j^{\sigma\tau}$ be an isomorphism. Then θ is a permutation of $V \cup \{\infty\} \equiv V'$. If $\theta : k \rightarrow \infty$, where $k \neq \infty$, then from Fig. 2, we see that θ must map the anti-derivative $D_i(\bar{k})$ to the derivative $\bar{D}_j^{\sigma\tau}(\theta(\infty))$, which is impossible, since D_i and D_j are compatible designs. So only $\theta(\infty) = \infty$ is possible. It follows that $D_i^{\theta} = D_i$, i.e., $\theta \in \mathrm{Aut}(D_i)$ □

Corollary 7. *Let D_i and D_j be compatible designs. Then* $\mathrm{Aut}(D_i\bar{D}_j)^{\sigma} = \mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j)$.

Theorem 8. *Let D_i and D_j be compatible $(4t+3, 2t+1, t)$-SBIBD's with the same variety set V, and let $\sigma \in \mathrm{Sym}(V)$. Then the number of $\tau \in \mathrm{Sym}(V)$ such that $D_i\bar{D}_j^{\sigma}$ and $D_i\bar{D}_j^{\sigma\tau}$ are isomorphic is*

$$\frac{|\mathrm{Aut}(D_i)| \cdot |\mathrm{Aut}(D_j)|}{|\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})|}.$$

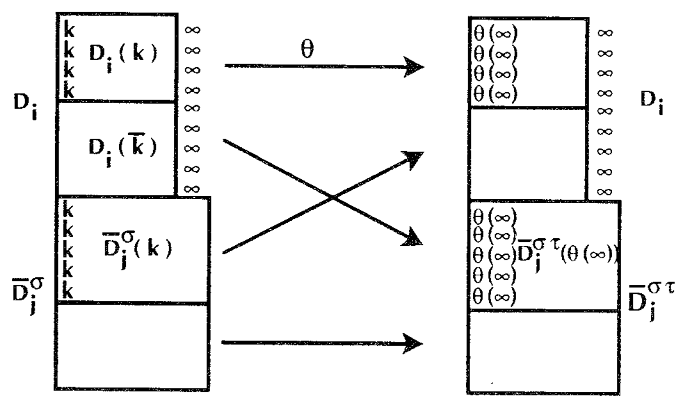

Fig. 2.

Proof. Let $\theta : D_i\bar{D}_j^{\sigma} \to D_i\bar{D}_j^{\sigma\tau}$ be an isomorphism. From Lemma 6, we know that $\theta \in \mathrm{Aut}(D_i)$. It follows that $D_j^{\sigma\theta} = D_j^{\sigma\tau}$, or in other words, $\tau\theta^{-1} \in \mathrm{Aut}(D_j^{\sigma})$, so that $\tau \in \mathrm{Aut}(D_j^{\sigma})\theta$. Now two cosets $\mathrm{Aut}(D_j^{\sigma})\theta_1$ and $\mathrm{Aut}(D_j^{\sigma})\theta_2$ are equal if and only if $\theta_1\theta_2^{-1} \in \mathrm{Aut}(D_j^{\sigma})$. But since $\theta_1,\ \theta_2 \in \mathrm{Aut}(D_i)$, we conclude that $\theta_1\theta_2^{-1} \in \mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})$. Hence, for each coset of $\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})$ in $\mathrm{Aut}(D_i)$, there are $|\mathrm{Aut}(D_j^{\sigma})|$ values of τ which give isomorphic designs. So for each $\sigma \in \mathrm{Sym}(V)$, there are

$$\frac{|\mathrm{Aut}(D_i)|\cdot|\mathrm{Aut}(D_j)|}{|\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})|}$$

designs $D_i\bar{D}_j^{\sigma\tau}$ which are isomorphic to $D_i\bar{D}_j^{\sigma}$. □

We now state and prove our two main theorems.

Theorem 9. *If D_i and D_j are compatible* $(4t+3,\ 2t+1,\ t)$*-SBIBD's, then there are at least*

$$\left\lceil \frac{(4t+3)!}{|\mathrm{Aut}(D_i)|\cdot|\mathrm{Aut}(D_j)|} \right\rceil$$

non-isomorphic $(4t+4,\ 8t+6,\ 4t+3,\ 2t+2,\ 2t+1)$*-BIBD's.*

Proof. There are $(4t+3)!$ permutations $\sigma \in \mathrm{Sym}(V)$. By Theorem 7, there are

$$\frac{|\mathrm{Aut}(D_i)|\cdot|\mathrm{Aut}(D_j)|}{|\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})|}$$

permutations $\tau \in \mathrm{Sym}(V)$ for which $D_i\bar{D}_j^{\sigma\tau}$ is an isomorphic design. In general, we do not know $|\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j^{\sigma})|$, which could be as small as 1. This gives at least

$$\frac{(4t+3)!}{|\mathrm{Aut}(D_i)|\cdot|\mathrm{Aut}(D_j)|}$$

choices for σ which produce mutually non-isomorphic designs. □

We will denote by $\{D_i\bar{D}_j\}$ a set of mutually non-isomorphic $(4t+4,\ 8t+6,\ 4t+3,\ 2t+2,\ 2t+1)$-BIBD's of the form $D_i\bar{D}_j^{\sigma}$ with as many designs as possible.

The previous theorem proves that there are many non-isomorphic $(4t+4,\ 8t+6,\ 4t+3,\ 2t+2,\ 2t+1)$-BIBD's. The next theorem shows that this number can be increased.

Theorem 10. *Let D_1, D_2, D_3, D_4 be* $(4t+3,\ 2t+1,\ t)$*-SBIBD's. If D_1 and D_3 are agreeable, or if D_2 and D_4 are agreeable, or if D_1 and D_4 are compatible, or if D_2 and D_3 are compatible, then $D_1\bar{D}_2$ is non-isomorphic to $D_3\bar{D}_4$.*

Proof. Assume that $D_1\bar{D}_2$ is isomorphic to $D_3\bar{D}_4$. Let θ be the isomorphism between them. If θ maps ∞ onto ∞, then $D_1 \cong D_3$. This contradicts the fact that D_1 and D_3 are agreeable. Hence θ maps ∞ in $D_1\bar{D}_2$ onto $k \neq \infty$ in $D_3\bar{D}_4$. As in the previous theorem, $D_1(\theta^{-1}(\infty))$ is isomorphic to $D_3(k)$ and $D_1(\overline{\theta^{-1}(k)}) \cong \bar{D}_4(k)$. Furthermore, $D_3(\bar{k}) \cong \bar{D}_2(\theta^{-1}(\infty))$, and since $\bar{D}_2(\overline{\theta^{-1}(\infty)}) \cong \bar{D}_4(\bar{k})$, then $D_1(\theta^{-1}(\infty)) \cong D_4(k)$. One of the four conditions of the theorem is violated. Hence $D_1\bar{D}_2$ can not be isomorphic to $D_3\bar{D}_4$. □

3. (15, 7, 3)-SBIBD's and (16, 30, 15, 8, 7)-BIBD's

Nandi [10] was the first to find that there are 5 non-isomorphic (15, 7, 3)-SBIBD's. We will use the designs listed in Bhat and Shrikande [4]. We will also use their numbering $D_1, D_2, \ldots, D_5$. In this section we will be using Kocay's [8] graph isomorphism program to find isomorphisms between designs. The algorithm is implemented on an Amdahl 580 and took only a few seconds in total CPU time. Table 1 relates the (15, 7, 3)-SBIBD's to the order of their automorphism groups.

Table 2 displays the order of the intersection $\mathrm{Aut}(D_i) \cap \mathrm{Aut}(D_j)$ of the automorphism groups of the pairs of (15, 7, 3)-SBIBD's.

Table 3 relates which $D_i(\bar{j})$ are isomorphic to which $\bar{D}_{i'}(j')$ for the (15, 7, 3)-SBIBD's. In order to shorten the table, we state here that in these particular designs, the design $D_i(\bar{j})$ with ∞ added to each block is always isomorphic to $\bar{D}_i(j)$. Hence we need only list the isomorphic $D_i(\bar{j})$. All designs in the same line of the table are isomorphic.

Next we list which $D_i(j)$ are isomorphic to which $D_{i'}(j')$ in order to tell us which designs are agreeable. Again, all designs listed on one line of the table are isomorphic.

We summarize the information in Table 3 and 4 with the following theorem.

Table 1

DesignD	| Aut(D) |
D_1	168
D_2	96
D_3	168
D_4	576
D_5	20160

Table 2

$\cap$	D_1	D_2	D_3	D_4	D_5
D_1	168	8	1	8	1
D_2	8	96	2	8	1
D_3	1	2	168	1	1
D_4	8	8	1	576	1
D_5	1	1	1	1	20160

Table 3

Iso. Class i	$D_1(\bar{i})$	$D_2(\bar{i})$	$D_3(\bar{i})$	$D_4(\bar{i})$	$D_5(\bar{i})$
1	1,2,6,7,9,11,12,14				
2	3,4,8,10,13,15	3,4,8,10,13		1-3,6-14	
3	5	15			
4		5	5	4,5,15	1-15
5		1,2,6,7,9,11	1-4,6-8,11,13		
6		12,14	9,10,12,14,15		

Table 4

Class i	$D_1(i)$	$D_2(i)$	$D_3(i)$	$D_4(i)$	$D_5(i)$
1	1,2,6,7,9,11,12,14	3,4,8,10,13,15		1-3,6-14	
2	3,4,5,8,10,13,15				
3		1,2,6,7,9,11,12,14	1-4,6-15		
4		5	5	4,5,15	1-15

Theorem 11. *D_1 and D_3, and D_1 and D_5 are the only pairs of agreeable* $(15, 7, 3)$*-SBIBD's. Also, they are the only compatible pairs of* $(15, 7, 3)$*-SBIBD's.*

Before proceeding, some history of $(16, 30, 15, 8, 7)$-BIBD's is necessary. Bhat and Shrikande [4] constructed 25 $(16, 30, 15, 8, 7)$-BIBD's by constructing $D_i\bar{D}_j$, for $i, j \in \{1, 2, 3, 4, 5\}$. They found that at least 21 of them were pair-wise non-isomorphic. However they made a mistake in calculating the characteristic number of $D_2\bar{D}_1$, which should have been 44. Hence their claim should have been that this list contains only 20 non-isomorphic designs. They also claimed that their designs were non-isomorphic to all of the 30 pair-wise non-isomorphic designs of Preece [11]. However, isomorphic copies of the design $D_5\bar{D}_5$ were in both lists. Hence the number of known non-isomorphic $(16, 30, 15, 8, 7)$-BIBD's at that time was 49.

Again using Kocay's program, it was found that only the following pairs in the list $D_i\bar{D}_j$ are isomorphic:

(1) $D_1\bar{D}_2$ and $D_2\bar{D}_1$,

(2) $D_1\bar{D}_4$ and $D_4\bar{D}_1$.

Hence there are 23 pairwise non-isomorphic $(16, 30, 15, 8, 7)$-BIBD's in Bhat and Shrikande's list.

Furthermore, the $D_i\bar{D}_i$ designs are the only possible resolvable $(16, 30, 15, 8, 7)$-BIBD's, and the only possible 3-$(16, 8, 7)$ designs, so we can improve the results in Mathon and Rosa [9] and Beth and Jungnickel [1] by the following theorems.

Theorem 12. *The number of non-isomorphic resolvable* $(16, 30, 15, 8, 7)$*-BIBD's is* 5.

Theorem 13. *The number of non-isomorphic resolvable* 3-$(16,8,7)$*-designs is* 5.

We are now ready to improve on the number of known non-isomorphic (16, 30, 15, 8, 7)-BIBD's.

Theorem 14. *The number of non-isomorphic* (16, 30, 15, 8, 7)*-BIBD's is at least* 92,436,200.

Proof. Consider the designs $H = \{D_1\bar{D}_3\} \cup \{D_1\bar{D}_5\} \cup \{D_3\bar{D}_1\} \cup \{D_5\bar{D}_1\}$. Since D_1 is compatible with D_3 and D_5, we can state, using Theorem 8, that the 4 sets are sets of mutually non-isomorphic designs of sizes

$$\frac{15!}{168 \cdot 168}, \quad \frac{15!}{168 \cdot 20160}, \quad \frac{15!}{168 \cdot 168}, \quad \text{and} \quad \frac{15!}{20160 \cdot 168}.$$

Since D_1 is also agreeable with D_3 and with D_5, we can use Theorem 9 to state that the number of non-isomorphic designs in H *is*

$$\frac{2 \cdot 15!}{28,224} + \frac{2 \cdot 15!}{3,386,880} = 92,436,200.$$

4. (19, 9, 4)-SBIBD's and (20, 38, 19, 10, 9)-BIBD's

The (19, 9, 4)-SBIBD's were enumerated by Gibbons [5]. There are six non-isomorphic (19, 9, 4)-SBIBD's. We will use the list and the numbering that occurs in Bhat [3]. Again using Kocay's program [8], we list several useful tables as in the previous section. Table 5 relates the (19, 9, 4)-SBIBD's with the order of their automorphism groups.

Table 6 displays the order of the intersection $\text{Aut}(D_i) \cap \text{Aut}(D_j)$ of the automorphism groups of D_i and D_j.

We mention here that D_1 and D_2 come from one Hadamard matrix, D_3 and D_4 from a different Hadamard matrix, and D_5 and D_6 from a third. Thus we note that if the intersection in Table 6 is nontrivial, then the designs are embedded in the same Hadamard matrix of order 20.

Table 5

DesignD	| Aut(D) |
D_1	171
D_2	9
D_3	72
D_4	8
D_5	24
D_6	6

Table 6

∩	D_1	D_2	D_3	D_4	D_5	D_6
D_1	171	9	1	1	1	1
D_2	9	9	1	1	1	1
D_3	1	1	72	8	1	1
D_4	1	1	8	8	1	1
D_5	1	1	1	1	24	2
D_6	1	1	1	1	2	6

Table 7

$D_i(\bar{j})$ $\cong$	$\bar{D}_i(j)$
1-19	38
20,23-26,28,30,35,36	21,22,27,29,31-34,37
21,22,27,29,31-34,37	20,23-26,28,30,35,36
38	1-19
39	39
40,42,46,48,50,52,54,56	59
41,43,45,47,49,51,53,55,57	60
58	58
59	40,42,44,46,48,50,52,54,56
60	41,43,45,47,49,51,53,55,57
61,63,65,67	61,63,65,71
62,64,66,72	68,70,74,76
67,69,73,75,97,98,99	67,69,73,75,97,98,99
68,70,74,76	62,64,66,72
77-80,82-85,87-90	96,100,105
81,86,91	81,86,91
92-95	110
96,100,105	77-80,82-85,87-90
101,102,106,107,111,112	103,104,108,109,113,114
103,104,108,109,113,114	101,102,106,107,111,112
110	92,93,94,95

Table 7 relates which $D_i(\bar{j})$ are isomorphic to which $\bar{D}_{i'}(j')$ for $(19,9,4)$-SBIBD's. In the table, all designs on the same line are isomorphic. In order to save space, $D_i(\bar{j})$ is represented by the integer $19(i-1)+j$ on the left side, and $\bar{D}_i(j)$ is represented by the integer $19(i-1)+j$ on the right side of the table.

We summarize the information in Tables 5–7 with the following theorem.

Theorem 15. *The following are the incompatible pairs of* (19, 9, 4)*-SBIBD's.*

D_1 with D_2,
D_2 with D_1 and D_2,
D_3 with D_3 and D_4,
D_4 with D_3, D_4, and D_6,
D_5 with D_5 and D_6,
D_6 with D_4, D_5, and D_6.

The other pairs are compatible.

Table 8 shows which $D_i(j)$ are isomorphic. Again, designs on the same line are isomorphic.

This information can be summarized in the following theorem.

Theorem 16. *The following* (19, 9, 4)*-SBIBD pairs are not agreeable.*

D_1 with D_2,
D_3 with D_4 and D_5,
D_4 with D_6,
D_5 with D_6.

All other pairs of designs are agreeable.

Table 8

Class	$D_1(i)$	$D_2(i)$	$D_3(i)$	$D_4(i)$	$D_5(i)$	$D_6(i)$
1	1-19	19				
2		1-18				
3			1			
4				1		
5			2-18 (evens)	2	5,10,15	
6			3-19 (odds)	3		
7				4-18 (evens)		2,3,4
8				5-19 (odds)		
9					1-4,6-9,11-14	1,5,10
10					16-19	15
11						6-9,11-14,16-19

Before proceeding, some history of the (20, 38, 19, 10, 9)-BIBD's is in order. Bhat [3] constructed the 36 designs $D_i\bar{D}_j$ and found that if $i \neq j$, then the designs were not isomorphic, except possibly for $D_5\bar{D}_6$ and $D_6\bar{D}_5$. Indeed, these two designs are not isomorphic. She also found that $D_1\bar{D}_1$ and $D_2\bar{D}_2$ were non-isomorphic to the other designs. This is also true for the pair $D_3\bar{D}_3$ and $D_4\bar{D}_4$, and for the pair $D_5\bar{D}_5$ and $D_6\bar{D}_6$. Again, she could not tell if the pairs of designs were isomorphic or not. It turns out that each pair is isomorphic, so that there are 33 non-isomorphic designs constructed in this fashion.

Again sharpening the Mathon and Rosa [9] tables, we state the following theorem.

Theorem 17. *The number of non-isomorphic resolvable* (20, 38, 19, 10, 9)-*BIBD's is* 3.

Applying Corollary 4, we get the following theorem.

Theorem 18. *The number of non-isomorphic* 3-(20, 10, 9)-*designs is* 3.

We are now ready to improve on the number of known non-isomorphic (20, 38, 19, 10, 9)-BIBD's.

Theorem 19. *The number of non-isomorphic* (20, 38, 19, 10, 9)-*BIBD's is at least*

10,869,917,004,894,316.

Proof. Consider

$$\{D_1\bar{D}_1\} \cup \{D_1\bar{D}_3\} \cup \{D_1\bar{D}_4\} \cup \{D_1\bar{D}_5\} \cup \{D_1\bar{D}_6\} \cup \{D_2\bar{D}_3\} \cup \{D_2\bar{D}_4\}$$
$$\cup \{D_2\bar{D}_5\} \cup \{D_2\bar{D}_6\} \cup \{D_3\bar{D}_1\} \cup \{D_3\bar{D}_2\} \cup \{D_4\bar{D}_1\} \cup \{D_4\bar{D}_2\} \cup \{D_5\bar{D}_1\}$$
$$\cup \{D_5\bar{D}_2\} \cup \{D_5\bar{D}_3\} \cup \{D_5\bar{D}_4\} \cup \{D_6\bar{D}_1\} \cup \{D_6\bar{D}_2\} \cup \{D_6\bar{D}_3\}.$$

Using Theorem 8, we see that each set $\{D_i\bar{D}_j\}$ has D_i and D_j compatible, and hence there are at least

$$\frac{19!}{|\mathrm{Aut}(D_i)|\cdot|\mathrm{Aut}(D_j)|}$$

non-isomorphic designs in each set. Applying Theorem 10, we see that no two designs from different sets are isomorphic. Therefore, the number of non-isomorphic (20, 38, 19, 10, 9)-BIBD's is at least

$$\begin{aligned}&\frac{19!}{171}\left(1+\frac{1}{72}+\frac{1}{8}+\frac{1}{24}+\frac{1}{6}\right)+\frac{19!}{9}\left(\frac{1}{72}+\frac{1}{8}+\frac{1}{24}+\frac{1}{6}\right)+\cdots\\ &=19!\,\frac{27871}{311904}\\ &=10{,}869{,}917{,}004{,}894{,}316. \qquad \square\end{aligned}$$

5. $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBD's, for $t=5, 6$, and 7

First, some history; Ito, Leon, and Longyear [7] found that there are at least 129 non-isomorphic (24, 46, 23, 12, 11)-BIBD's. Tonchev [11] found 4 non-isomorphic (28, 54, 27, 14, 13)-BIBD's, while Bhat and Shrikande [4] found 20 non-isomorphic (32, 62, 31, 16, 15)-BIBD's. Our approach to finding many more of these designs is to examine the cyclic $(4t+3,\ 2t+1,\ t)$-SBIBD's that are listed in the back of Hall's book [6]. We need then only check that he extended anti-derivative of the cyclic design is not isomorphic to the derivative of the complement of the design. If they are non-isomorphic, then Theorem 9 can be applied. This is how we obtain the following.

Theorem 20. *The number of non-isomorphic* (24, 46, 23, 12, 11)*-BIBD's is at least*

403,880,965,784,264,348.

Proof. Let D be the design with base block (1, 2, 3, 4, 5, 7, 9, 10, 13, 14, 17, 19) which is developed modulo 23. Then the extended anti-derivative of D is not isomorphic to the derivative of $\bar{D}$. Also, $|\mathrm{Aut}(D)|=23\times 11=253$. Apply Theorem 9. $\square$

Theorem 21. *The number of non-isomorphic* (28, 54, 27, 14, 13)*-BIBD's is at least*

9,820,329,245,540,352,000,000.

Proof. Let D be the design with base block

$$[(0,0,1),(1,0,0),(1,2,0),(1,1,1),(2,0,2),(1,1,0),(1,0,2),$$
$$(0,2,0),(0,2,1),(1,2,1),(2,1,1),(0,2,2),(2,2,1)],$$

which is developed modulo $(3,3,3)$. Then its extended anti-derivative is not isomorphic to the derivative of $\bar{D}$. Also $|\mathrm{Aut}(D)| = 27 \times 13 \times 3 = 1053$. Apply Theorem 9. □

Theorem 22. *The number of non-isomorphic* (32, 62, 31, 16, 15)*-BIBD's is at least*

$$38{,}029{,}083{,}844{,}041{,}728{,}836{,}746{,}735{,}484.$$

Proof. Let D be the design with base block (1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28), which is developed modulo 31. Then its extended anti-derivative is not isomorphic to the derivative of $\bar{D}$. Further, since $|\mathrm{Aut}(D)| = 31 \times 15$, we can apply Theorem 9 to obtain the result. □

These last three theorems lead us to ask the following question.

Question 23. Do there exist any cyclic $(4t+3, 4t+3, 2t+2, 2t+1, t)$-SBIBD's, where $t > 3$, whose extended anti-derivative is isomorphic to a derivative of the complement?

Comment. The restriction $t > 3$ is included, because for $t = 1, 2$, and 3, the extended anti-derivative of the cyclic design is isomorphic to the derivative of the complement of the design.

References

[1] T. Beth, D. Jungnickel and H. Lenz, Design Theory (Bibliographisches Institut, Mannheim, 1985).

[2] V.N. Bhat, Non-isomorphic solutions of some balanced incomplete block designs II, JCT(A) 12 (1972) 217–224.

[3] V.N. Bhat, Non-isomorphic solutions of some balanced incomplete block designs III, JCT(A) 12 (1972) 225–252.

[4] V.N. Bhat and S.S. Shrikande, Non-isomorphic solutions of some balanced incomplete block designs I, JCT(A) 9 (1970) 174–191.

[5] P.B. Gibbons, Computing techniques for the construction and analysis of block designs, Ph.D. Thesis, U. of Toronto, 1976.

[6] M. Hall, Jr, Combinatorial Theory, 2nd Ed. (Wiley, New York, 1986).

[7] N. Ito, J.S. Leon, and J.Q. Longyear, Classification of 3-(24, 12, 5) designs and 24-dimensional Hadamard matrices, JCT(A) 31 (1981) 66–93.

[8] W.L. Kocay, Abstract data types and graph isomorphism, J. Combin. Inform. System Sci. 9 (4) (1984) 247–259.

[9] R. Mathon and A. Rosa, Tables of parameters of BIBD's with $r \leq 41$, including existence, enumeration, and resolvability results, Ann. Discrete Math. 26 (North-Holland, Amsterdam, 1985) 275–308.

[10] H.K. Nandi, Enumeration of non-isomorphic solutions of balanced incomplete block designs, Sankhya 7 (1946) 305–312.

[11] D.A. Preece, Incomplete block designs with $v = 2k$, Sankhyā Ser. A 29 (1967) 305–316.

[12] V.D. Tonchev, Hadamard matrices of order 28 with an automorphism of order 13, JCT(A) 35 (1983) 43–57.