Covering Designs on 13 Blocks Revisited

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Abstract

A (v, k, t)-Covering Design is a set of k-subsets of a set of v elements such that every set of t elements occurs in at least one block. The minimum number of blocks in any such design is denoted by $C(v, k, t)$. We find all v and k so that $C(v, k, 2) = 13$ and give all the details and proofs. Further, we determine that $C(28, 9, 2) = C(41, 13, 2) = 14$.

1 Introduction

A (v, k, t)-Covering Design is a set of k-subsets (called blocks) of a set of v elements such that every set of t elements occurs in at least one block. The minimum number of blocks in any such design is denoted by $C(v, k, t)$. Here is a $(7, 3, 2)$-Covering Design: $\{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 4\}, \{7, 3, 5\}, \{1, 3, 5\}\}$. It is not minimal as it is well-known that $C(7, 3, 2) = 7$. In 1979, Mills [4] found all v and k, where $C(v, k, t) = m$ for $m = 1, 2, \ldots, 12$. In 1985, Todorov [6] found all v and k where $C(v, k, 2) = 13$, with two possible exceptions. However, in the 4 pages allotted to him, Todorov could not give all the proofs. This paper is

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very difficult to obtain and his thesis which contains all the proofs is even harder to obtain; in fact, we could not obtain it. This effectively killed off all research in this direction. Both the survey paper by Mills and Mullin [5] and the article by Stinson in the CRC Handbook of Combinatorial Designs [10] ignore the result even though the paper was reviewed by Stanton [9], and the results were also cited by Todorov [8]. So we will fill in all the missing proofs following the trail blazed by Todorov.

Of course, some of our proofs differ in detail as we have a stronger result that includes both of Todorov's two possible exceptions, namely that $C(41, 13, 2) = 14$ and $C(28, 9, 2) = 14$. We also give an easy proof that the infinite family of $(13r + 3, 4r + 1, 2)$-Covering Designs can not exist on 13 blocks. This proof can be generalized to other situations.

A key technique in constructing covering designs is taking a covering design with small v and k and replacing each original element, i, by α_i new elements. Elements are then added to the shorter blocks to make all block sizes equal. For the problem of ascertaining whether $C(v, k, 2)$=13, one particular construction deals with most cases. This construction has been formalized by Todorov [6].

Theorem 1.1 *If $a_0 \geq a_1 \geq a_2 > 0$ and p is a prime power, then*

$$C(a_0p^2 + a_1p + a_2, a_0p + a_1, 2) = p^2 + p + 1.$$

Proof: Replace an arbitrary point, x, in a projective plane of order p by a_2 new ones and replace every other point on some line containing the original point x with a_1 new points. Finally, replace all other points in the plane with a_0 new points. Interpret the result as a covering design and fill up the short blocks. ∎

For $a_2 > 0$, this construction attains the well-known Schönheim bound which we now state.

Theorem 1.2 $C(v, k, t) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t-1} \right\rceil \cdots \right\rceil \right\rceil$.

Using Mills [4], it is easy to determine that if we are interested in $C(v, k, 2) = 13$, we need only consider v's in the range $3k < v \leq 13k/4$. Using $p = 3$, Theorem 1.1 actually determines $C(v, k, 2) = 13$ for most of the cases where $3k < v \leq 13k/4$. To see this, let $4 \leq k = 4x + j$ and take $a_0 = x$ if $j = 0$ and $a_0 = x + 1$ if $1 \leq j \leq 3$, then take $a_1 = k - 3a_0$ and $a_2 = v - 9a_0 - 3a_1$. This leaves three infinite families where $a_2 \leq a_1$ fails:
1) $(13r + 2, 4r + 1, 2)$-Covering Designs,
2) $(13r + 3, 4r + 1, 2)$-Covering Designs,
3) $(13r + 6, 4r + 2, 2)$-Covering Designs.

Some of the smaller covering numbers have been determined and are in Gordon's Tables [2]. In particular, $C(15, 5, 2) = 13$ and $C(16, 5, 2) = 15$. Note that, although $v = 15$, $k = 5$ apparently falls in the first series, here $v/k \not> 3$ for $r = 1$. Bate, Li and van Rees [1] proved that $C(19, 6, 2) = 15$.

These three values all exceed the Schönheim bound. We now develop an easy way to handle the second infinite family.

Henceforth, we will only consider $t = 2$, i.e., pair coverings, and feel free to use t as a variable for other purposes.

2 (13r+3, 4r+1, 2)-Covering Designs

In this section, we develop equations that will quickly allow us to determine that this family can not exist on 13 blocks. Todorov proved the theorem for $r = 1, 2, 3$ explicitly. This, along with his result that $(13r+2, 4r+1, 2)$-Covering Designs require 14 blocks for $r \geq 4$, imply the following theorem for $r \geq 1$.

Theorem 2.1 (Todorov) $C(13r + 3, 4r + 1, 2) \geq 14$.

Proof: Assume that a $(13r + 3, 4r + 1, 2)$-Covering Design exists on 13 blocks. Since an element that appears in at most three blocks can occur with at most $3 \cdot 4r = 12r$ of the $13r+2$ other elements, every element in the covering design must occur at least 4 times. Since there are $13(4r + 1) = 52r + 13$ incidences in the covering design, one element must occur in 5 blocks and the remaining $13r + 2$ elements occur in 4 blocks. This implies that there is a block, B_0, that contains only elements of frequency 4, i.e., B_0 contains only elements that occur in exactly 4 blocks in the covering design. Let x_i be the number of other blocks that intersect B_0 in i elements.

By counting the blocks we get:

$$\sum_i x_i = 12.$$

By counting elements in an intersection we get:

$$\sum_i ix_i = 3 \cdot (4r + 1) = 12r + 3.$$

Manipulating the above two equations we get:

$$\sum_i 3r(i - r)x_i = 9r. \tag{1}$$

Let $b_i(x)$ be the number of blocks that an element, x, which is not in B_0, occurs in with exactly i elements from B_0. If x is an element of frequency 4 not in B_0, then we get two equations. The first counts the number of blocks x is in and the second counts the number of elements in B_0 that x has to occur with:

$$\sum_i b_i(x) = 4$$

and

$$\sum_i ib_i(x) \geq 4r + 1.$$

Then

$$\sum_i (i-r)b_i(x) \geq 1.$$

Let y be the element of frequency 5. Then we get the same kind of first equation but the second equation counts the number of elements in B_0 that y has to occur with by noting that this number is at least $5r$ if the 5 blocks containing y all intersect B_0 in at least r elements. If there are intersections smaller than size r then this original number would be smaller. We get:

$$\sum_i b_i(y) = 5$$

and

$$\sum_i ib_i(y) \geq 5r - x_{r-1} - 2x_{r-2} - 3x_{r-3} \cdots.$$

Then

$$\sum_i (i-r)b_i(y) \geq -x_{r-1} - 2x_{r-2} - 3x_{r-3} \cdots.$$

There are $(13r+3)-(4r+1) = (9r+2)$ elements not in B_0. One of these elements has frequency 5 and $9r+1$ of them have frequency 4 so:

$$\sum_{x \notin B_0} \sum_i (i-r)b_i(x) \geq 9r + 1 - x_{r-1} - 2x_{r-2} - 3x_{r-3} \cdots.$$

If we interchange the order of summation and note that $\sum_{x \notin B_0} b_i(x) = (4r+1-i)x_i$ we get:

$$\sum_i (i-r)(4r+1-i)x_i \geq 9r+1 - x_{r-1} - 2x_{r-2} - 3x_{r-3} \cdots. \quad (2)$$

If we subtract equation (1) from equation (2) we get

$$\sum_i -(i-r)(i-(r+1))x_i \geq 1 - x_{r-1} - 2x_{r-2} - 3x_{r-3} \cdots. \quad (3)$$

We may note the right hand side of equation (3) is $1 - \sum_{i=0}^{i=r-1}(r-i)x_i$, so equation (3) can be written as:

$$\sum_{i=0}^{r-1} -(i-r)^2 x_i + \sum_{i=r}^{4r+1} -(i-r)(i-(r+1))x_i \geq 1.$$

This is a contradiction as the left hand side of the inequality is negative. ■

3 (13r+6, 4r+2, 2)-Covering Designs

In this section we will assume that a $(13r+6, 4r+2, 2)$-Covering Design exists on 13 blocks. We will then explore what properties the blocks and elements have in such a posited covering. Eventually, we will get a contradiction that rules out such a covering on 13 blocks. The first lemma examines block intersection sizes for a block which only contains elements of frequency 4.

Lemma 3.1 *In a $(13r+6, 4r+2, 2)$-Covering Design on 13 blocks, the block intersection sizes for a block containing only frequency 4 elements are $r-2$, $r-1$, r, $r+1$ or $r+2$. Further in this case, there is at most one block intersection of size $r-2$ or $r+2$.*

Proof: Assume that a $(13r+6, 4r+2, 2)$-Covering Design exists on 13 blocks. Let B_0 be a particular block in the covering. Let x_i be the number of blocks that intersect B_0 in i elements. Then, as in Theorem 2.1, we can develop the following equations.

$$\sum_i x_i = 12$$

$$\sum_i i x_i = 12r + 6$$

$$\sum_i (3r+1)(i-r)x_i = 18r + 6.$$

Each element in this covering must appear in at least 4 blocks. This means that two elements appear in 5 blocks and the remaining $13r+4$ elements appear in 4 blocks or one element appears in 6 blocks and the remaining $13r+5$ appear in 4 blocks. Clearly, at least one block contains only elements of frequency 4. Let B_0 be such a block. Then, for $x \notin B_0$, let $b_i(x)$ be the number of blocks that x appears in with exactly i elements from B_0. We now develop 3 sets of equations for the elements of frequency 4, 5 and 6 similar to the equation in the latter part of Theorem 2.1.

$f = 4$:
$\sum b_i(x) = 4$
$\sum i b_i(x) \geq 4r + 2$
$\sum (i-r) b_i(x) \geq 2$

$f = 5$:
$\sum b_i(x) = 5$
$\sum i b_i(x) \geq 5r - x_{r-1} - 2x_{r-2} - \cdots$
$\sum (i-r) b_i(x) \geq -x_{r-1} - 2x_{r-2} - \cdots$

$f = 6$:
$\sum b_i(x) = 6$

$\sum ib_i(x) \geq 6r - x_{r-1} - 2x_{r-2} - \cdots$
$\sum (i-r)b_i(x) \geq -x_{r-1} - 2x_{r-2} - \cdots$

If we have two elements of frequency 5 and $9r+2$ elements of frequency 4, then we get:

$$\sum_{x \notin B_0} \sum_i (i-r)b_i(x) \geq 18r+4-2(x_{r-1}+2x_{r-2}\cdots) = 18r+4-\sum_{i=0}^{i=r-1} 2(r-i)x_i$$

or

$$\sum_{i=0}^{r-1} -(i-r)(i-(r-1))x_i + \sum_{i=r}^{4r+2} -(i-r)(i-(r+1))x_i \geq -2.$$

This implies that $x_i = 0$ except for $x_{r-2}, \ldots, x_{r+2}$ and implies that $x_{r-2} + x_{r+2} \leq 1$. The case of an element of frequency 6 is quite similar to the proof of Theorem 2.1. We get the equation:

$$\sum_{i=0}^{r-1} -(i-r)^2 x_i + \sum_{i=r+2}^{4r+2} -(i-r)(i-(r+1))x_i \geq 0.$$

This implies that $x_i = 0$ except for x_r and x_{r+1}. ■

The next two lemmas examine elements of frequency 4 in an intersection of blocks. But first we define t to be the size of the largest intersection of blocks in the covering.

Lemma 3.2 (Todorov [6]) *In a $(13r+6, 4r+2, 2)$-Covering Design on 13 blocks, assume that two blocks, B_i and B_j, intersect in $t_{i,j}$ elements where $t - t_{i,j} \leq r$. Then no two elements of frequency 4 in the intersection occur together exactly one more time.*

Proof: Assume that elements 1 and 2 have frequency 4 and occur in the intersection of B_i and B_j. Further assume that they do occur together in B_1 and then do not occur together again in any other block. Then this is the situation:
$B_i : \{1, 2, \ldots, t_{i,j}, (t_{i,j}+1), \ldots, (4r+2)\}$
$B_j : \{1, 2, \ldots, t_{i,j}, (4r+3), \ldots, (8r+4-t_{i,j})\}$
$B_1 : \{1, 2, \ldots\}$
$B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$
and there are no more occurrences of 1 or 2 in the covering. Let $X_2 = X \setminus (B_i \cup B_j)$. Then B_1 can contain at most $4r$ of the $(13r+6) - (8r+4-t_{i,j}) = 5r+2+t_{i,j}$ elements in X_2. Since the elements of X_2 must appear at least once with the elements 1 and 2, then both B_2 and B_3 must contain at least $5r+2+t_{i,j} - 4r = r+2+t_{i,j}$ of the X_2 elements. Then $|B_2 \cap B_3| \geq r+2+t_{i,j} \geq t+2$, which contradicts t being the largest intersection size. ■

Lemma 3.3 is a technical lemma that we use to give upper bounds on the number of pairs of some of the elements. The left hand inequality is tighter, but the right hand inequality is sometimes more convenient to manipulate. When we use Lemma 3.3, a_i is the number of elements in block i, which is unknown, and $\overline{a}$ is the average of the a_i, which is known.

Lemma 3.3 *Let a_i for $i = 1, 2, \ldots, n$, c and d be non-negative integers with $d < n$ and $\sum a_i = n\overline{a} = nc + d$. Then*

$$\sum \binom{a_i}{2} \geq (n-d)\binom{c}{2} + d\binom{c+1}{2} \geq \frac{n\overline{a}(\overline{a}-1)}{2}.$$

Proof: This follows by simple manipulation. The difference in the first inequality is $\sum(a_i - c)(a_i - c - 1)/2$, and the difference in the second inequality is $d(n-d)/(2n)$. ■

For the next lemma we need the following definition. Let ϵ be the the number of elements that occur with an element of frequency 4, including multiplicities, minus the number of other elements in the design, i.e., this is the excess of elements that a frequency 4 element occurs with considering that it needs to only occur with another element once. In these coverings $\epsilon = 4(4r+1) - (13r+5) = 3r - 1$. In the excess multigraph, the vertices are the elements of the covering, and the number of edges between any two elements is one less than the number of blocks containing the pair. If an element has frequency $4 + y$ in the covering design, then the degree of that element in the (loop-free) multigraph is $(4+y)(k-1) - (v-1)$.

Lemma 3.4 (Todorov [6]) *In a $(13r+6, 4r+2, 2)$-Covering Design on 13 blocks with $r \geq 1$, assume that two blocks, B_i and B_j intersect in $t_{i,j}$ elements where $t - t_{i,j} \leq r$ and such that $t_{i,j} \geq r+1$. Then any 2 elements of frequency 4 in the intersection of B_i and B_j must intersect again.*

Proof: Let $t_{i,j} = r + 1 + m$, with $m \geq 0$. Assume that elements 1 and 2 have frequency 4 and occur in the intersection of B_i and B_j. Further assume that they do not occur together again in any other block. Then this is the situation:

$B_i : \{1, 2, \ldots, r+1+m, r+2+m, \ldots, (4r+2)\}$
$B_j : \{1, 2, \ldots, r+1+m, (4r+3), \ldots, 7r+3-m\}$
$B_1 : \{1, \ldots\}$ $B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$ $B_4 : \{2, \ldots\}$

As before let $X_2 = X \setminus (B_i \cup B_j)$, so $|X_2| = 13r + 6 - (7r + 3 - m) = 6r + 3 + m$. The total frequency of the X_2 elements is $4 \cdot (6r + 3 + m) + x$ where $0 \leq x \leq 2$.

Every X_2 element must occur at least once in B_1 and B_2, and at least once in B_3 or B_4. To minimize the number of X_2 vs. X_2 pairs in the above blocks we must have that the X_2 elements occur as equally often as possible in blocks B_1 and B_2 and in blocks B_3 and B_4. The remaining occurrences of the elements of X_2 must be distributed as evenly as possible over the last 7 blocks. The average number of X_2 elements in blocks B_1 through B_4 exceeds the average number of X_2 elements in the last seven blocks by over 1, so we want to maximize the number of X_2 incidences in the last seven blocks to reduce the total number of X_2 vs. X_2 pairs.

Now, using Lemma 3.3, we can bound the pairs generated by I incidences in b blocks by $(I/b-1)I/2$. For the $2(6r+3+m)$ incidences in blocks B_1 through B_4 this bound is:

$$(36r^2+24r+12rm+4m+m^2+3)/2.$$

For the $(12r+6+2m+x)$ incidences in the last seven blocks this bound is:

$$(144r^2+60r+48rm-6+10m+4m^2+x(24r+5+4m+x))/14.$$

So the total number of X_2 vs. X_2 pairs is bounded below by:

$$(396r^2+132rm+228r+38m+11m^2+15+24xr+5x+4mx+x^2)/14.$$

Now the maximum number of X_2 vs. X_2 pairs occurs if every X_2 element in the excess multigraph is joined to another X_2 element, and the total degree of these elements is $|X_2|\cdot(3r-1)+x(4r+1)$, and we halve this for edges/pairs, and then we need to add $\binom{|X_2|}{2}$ (the non-excess pairs). This total number evaluates to:

$$(54r^2+15rm+33r+4m+m^2+3+4xr+x)/2,$$

and the difference is

$$(18r^2+27rm-3r+10m+4m^2-6-4xr-2x+4mx+x^2)/14.$$

Rewriting this difference as $((r-1)(18r+27m+15-4x)+(3-x)^2+37m+4mx+4m^2)/14$, we clearly see the difference is positive, so we must generate too many X_2 pairs. ■

We now have the lemmas to eliminate the $(13r+6,4r+2,2)$-Covering Designs on 13 blocks.

Theorem 3.5 $C(13r+6,4r+2,2)>13$.

Proof: Clearly $C(6,2,2)=15$, so let $r\geq 1$. Assume there exists a $(13r+6,4r+2,2)$-Covering Design on 13 blocks where t, the size of the largest intersection of blocks, is greater than or equal to $r+3$. Also assume that the number of elements in the intersection of B_i and B_j is $t_{i,j}=t=r+3+m$, with $m\geq 0$. This intersection obeys all the conditions of Lemmas 3.2 and 3.4. So the only way that the elements of frequency 4 (there are at least $r+3+m-2=r+1+m$ of them) can intersect is as follows:
$B_i:\{1,2,\ldots,r+3+m,\ldots\}$
$B_j:\{1,2,\ldots,r+3+m,\ldots\}$

$B_1 : \{1, \ldots, r+1+m, \ldots\}$
$B_2 : \{1, \ldots, r+1+m, \ldots\}$
Then, calculating the excess of element 1, we get $\epsilon \geq r+2+m+2(r+m) = 3r+2+2m > 3r-1$ which is a contradiction. So $t \leq r+2$. Now there is a block, B_0, containing only elements of frequency 4. By Lemma 3.1 and averaging, we know that B_0 must intersect a block, say B_{00}, in at least $r+1$ elements. Since $t - t_{i,j} \leq r+2-(r+1) = 1$, this intersection also obeys all the conditions of Lemmas 3.2 and 3.4. So the elements in this intersection all occur together in 4 blocks. This means the excess of any of these elements is at least $3r > 3r-1$, a contradiction. ∎

4 (13r+2,4r+1,2)-Covering Designs

In this section we will assume that a $(13r+2, 4r+1, 2)$-Covering Design exists on 13 blocks. We will then explore what properties the blocks and elements have in such a posited covering. This will be long and detailed, following the lemmas and definitions used by Todorov [6]. The main idea is that such a covering design had to have come from the "blow-up" of the projective plane. By examining the projective plane on 13 points, we will show that the covering design could not have come from this plane by "blowing up" its points. Hence the covering design on 13 blocks does not exist.

The start of this section goes through the same kind of theorems as in the previous section. By elementary counting, we know that there are at least $13r-3$ elements of frequency 4 and as many as 5 elements of frequency greater than 4. The pairs excess, ϵ, for an element of frequency $4+x$ is $3r-1+x(4r) = (3+4x)r-1$. Let t be the size of the maximum intersection between blocks of the covering. The first lemma of this section is essentially the same as Lemma 3.2 and so the proof will not be given.

Lemma 4.1 (Todorov [6]) *In a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks with $r \geq 2$, assume that two blocks, B_i and B_j, intersect in $t_{i,j}$ elements where $t - t_{i,j} \leq r$. Then no two elements of frequency 4 in the intersection occur together exactly one more time.*

Although Lemma 4.1 holds for $r = 2$, we also need to establish that Lemma 4.1 will apply to any case where $t_{i,j} = r+1$ when $r = 2$, and we can do this be limiting the size of t. We use the exponential notation a^b to mean b blocks containing a elements each.

Lemma 4.2 *In a $(28, 9, 2)$-Covering Design on 13 blocks, if $|B_i \cap B_j| = t_{i,j}$, then $t_{i,j} \leq 5$.*

Proof: If two blocks, say B_i and B_j, intersect in $t_{i,j} \geq 6$ elements, then these two blocks contain $2(k-t_{i,j})$ unrepeated elements and $t_{i,j}$ repeated elements, so there are $v-2k+t_{i,j}$ elements not in $B_i \cup B_j$. Since $t_{i,j} \geq 6$,

$B_i \cap B_j$ contains an element, say 1, of frequency 4, so the other two blocks that contain 1 must also contain the $v - 2k + t_{i,j}$ elements not in $B_i \cup B_j$, so $v - 2k + t_{i,j} + 2 \leq 2k$, i.e., $t_{i,j} \leq 6$.

Now suppose that B_i and B_j intersect in $t_{i,j} = 6$ elements and suppose that an element, u, of frequency 4 lies in blocks B_i, B_j, B_1 and B_2. We see that B_i and B_j together contain 12 distinct elements, so B_1 and B_2 each contain 8 of the remaining 16 elements. Consider the 8 elements of $B_1 \setminus \{u\}$, and suppose their total frequency is $32 + y$. Then these 8 elements have at most $28 + 8\epsilon/2 + y(k-1)/2 = 48 + 4y$ pairs amongst themselves. However, we must generate at least $49 + 2y$ pairs (the most equitable distribution of the elements is $8^1 0^3 2^{3-y} 3^{6+y}$ for $y \leq 3$), so we must have $y \geq 1$, and similarly $B_2 \setminus \{u\}$ contains a (disjoint) element of frequency 5 or more, so $B_i \cap B_j$ has at most 3 elements of frequency 5 or more. Repeating the above argument we get two more pairs of blocks with each pair covering the 16 elements of $(B_i \cup B_j)^c$. Now the most equitable distribution for $B_1 \setminus \{u\}$ (of $8^1 0^3 4^4 2^{2-y} 3^{3+y}$ for $y \leq 2$) generates $63 + 2y$ pairs, and $8^1 0^3 4^4 3^{7-y} 4^{y-2}$ for $y \geq 2$ generates $61 + 3y$ pairs, so we generate too many of these pairs and must have $t < 6$. ■

Lemma 4.3 is essentially the same as Lemma 3.4 given earlier. The proof differs in the detailed analysis so that will be supplied. The proof for the case $r = 2$ involves eliminating 80 subcases, often by ad hoc arguments, so we will defer the full proof for $r = 2$ to a later section. This lemma is stronger (it covers $r \geq 2$) than the corresponding lemma given by Todorov [6].

Lemma 4.3 (Todorov [6]) *In a $(13r + 2, 4r + 1, 2)$-Covering Design on 13 blocks with $r \geq 2$, assume that two blocks, B_i and B_j, intersect in $t_{i,j}$ elements where $t_{i,j} \geq r + 1$. Then any 2 elements of frequency 4 in the intersection of B_i and B_j must intersect again.*

Proof: Let $t_{i,j} = r + 1 + m$, with $m \geq 0$. Assume that elements 1 and 2 have frequency 4 and occur in the intersection of B_i and B_j. Further assume that they do not occur together again in any other block. Then this is the situation:

$B_i : \{1, 2, \ldots, r + 1 + m, r + 2 + m, \ldots, (4r + 1)\}$
$B_j : \{1, 2, \ldots, r + 1 + m, (4r + 2), \ldots, 7r + 1 - m\}$
$B_1 : \{1, \ldots\}$
$B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$
$B_4 : \{2, \ldots\}$

The proof proceeds as in the proof of Lemma 3.4. Let $X_2 = X \setminus (B_i \cup B_j)$, so $|X_2| = 6r + 1 + m$. Assume the total frequency of the X_2 elements is $4 \cdot (6r + 1 + m) + x$ where $0 \leq x \leq 5$.

Every X_2 element must occur at least once in B_1 and B_2, and at least once in B_3 or B_4. To minimize the number of X_2 vs. X_2 pairs in the above blocks we must have that the X_2 elements occur as equally often as possible in blocks B_1 and B_2 and in blocks B_3 and B_4. The remaining occurrences of the elements of X_2 must be distributed as evenly as possible over the last 7 blocks. The average number of X_2 elements in blocks B_1 through B_4 exceeds the average number of X_2 elements in the last seven blocks by over 1, so we want to maximize the number of X_2 incidences in the last seven blocks to reduce the total number of X_2 vs. X_2 pairs.

Now, using Lemma 3.3, we can bound the pairs generated by I incidences in b blocks by $(I/b-1)I/2$. For the $2(6r+1+m)$ incidences in blocks B_1 through B_4 this bound is:

$$(36r^2+12rm+m^2-1)/2.$$

For the $(12r+2+2m+x)$ incidences in the last seven blocks this bound is:

$$(144r^2-36r+48rm-10-6m+4m^2+x(24r-3+4m+x))/14.$$

So the total number of X_2 vs. X_2 pairs is bounded below by:

$$(396r^2+132rm-36r-6m+11m^2-17+24xr-3x+4mx+x^2)/14.$$

Now the maximum number of X_2 vs. X_2 pairs occurs if every X_2 element in the excess multigraph is joined to another X_2 element, and the total degree of these elements is $|X_2|\cdot(3r-1)+x(4r)$, and we halve this for edges/pairs, and then we need to add $\binom{|X_2|}{2}$ (the non-excess pairs). This total number evaluates to:

$$(54r^2+15rm+3r+m^2-1+4xr)/2,$$

and the difference is

$$(18r^2+27rm-57r-6m+4m^2-10-4xr-3x+4mx+x^2)/14.$$

The derivative with respect to m is $27r-6+8m+4x$ which is positive for $r\geq 1$, so the minimum for $m\geq 1$ occurs at $m=1$ and this minimum is $(18r^2-30r-4xr-12+x+x^2)/14$. Rewriting this minimum difference as $((r-3)(18r+24-4x)+(6-x)(5-x)+30)/14$, we clearly see the difference is positive, so we must generate too many X_2 pairs if $m\geq 1$. Now consider $m=0$, where the difference is $(18r^2-57r-10-4xr-3x+x^2)/14$ which can be rewritten as $((r-5)(18r+33-4x)+(11-x)(12-x)+23)/14$ which is clearly positive for $r\geq 5$. This difference is always negative for $1\leq r\leq 3$, and also for $r=4$ with $x\geq 4$.

So we need only prove the cases where $r\leq 4$ and $m=0$ and there are x extra occurrences of X_2 elements. If we use the stricter inequality in Lemma 3.3, then we find that the minimum number of X_2 vs. X_2 pairs generated is $108+3x$ for $x\leq 2$ and $106+4x$ for $x\geq 2$ when $r=2$; $247+5x$ for $x\leq 4$ and $243+6x$ for $x\geq 4$ when $r=3$; $442+7x$ for $x\leq 5$ when $r=4$. The corresponding maximum number permitted is $110+4x$ when

$r = 2$; $247 + 6x$ when $r = 3$; $437 + 8x$ when $r = 4$. (When r is even there is also at least one X_2 vs. X_2^c pair, since then the elements/vertices of X_2 all have odd degree in the excess multigraph, and $|X_2|$ is odd.)

Now if $r = 4$, then we generate too many pairs, except possibly if $x = 5$, but in this case we know the distribution of the X_2 elements over the last 11 blocks must be $13^2 12^2 7^1 8^6$, so now an element, α, of X_2 with frequency $5 + y$ (with $0 \leq y \leq 4$) exists (since $x = 5$) and it has at least $2(17 - 13) + (3 + y)(17 - 8) = 35 + 9y$ pairs with the 29 elements of X_2^c. Now α occurs with $16(5 + y) = 80 + 16y$ pairs of which at least $11 + 24 + 16(y + 1) - 1 = 50 + 16y$ are X_2 vs. X_2^c pairs. The minus one accounts for the special case that occurs when r is even. This leaves at most $80 + 16y - (50 + 16y) = 30$ X_2 vs. X_2 pairs. Thus, $r \neq 4$.

Now consider $r = 3$ with $m = 0$. Let the total frequency of the elements of X_2 be $4 \cdot 19 + x$. We can have at most $247 + 6x$ pairs amongst the 19 X_2 elements, but, of the 20 elements in $X_1 = X_2^c \setminus \{1, 2\}$, at least $20 - (5 - x) = 15 + x$ elements of X_1 have frequency 4, and they have appeared at least once in $B_i \cup B_j$, so each must appear with at least one block which contains at least $|X_2|/3$ (i.e., at least 7) elements of X_2. Since we can accommodate at most 5 elements of X_1 in $B_1 \cup B_2$, and at most 5 elements of X_1 in $B_3 \cup B_4$, and 6 elements of X_1 with 7 elements of X_2, we will need at least one block with 7 (or more) elements of X_2 (aside from the first six blocks), and at least two such blocks when $x \geq 2$. If all the excesses of X_2 elements are used to form X_2 vs. X_2 pairs, then we will have $247 + 6x$ X_2 vs. X_2 pairs. Even if we ignore the forcing of blocks with 7 or more X_2 elements, we will get to within $\min(x, 4)$ pairs of this number. We cannot force a block with 7 X_2 elements for $x = 0$, as this would generate too many pairs. Nor can we put an extra incidence on the first six blocks, as this not only increases the generated pair count, but also introduces an extra incidence with either element 1 or 2, and so reduces the allowed pair count. In Table 1, we list all feasible patterns of X_2 incidences on the last 11 blocks.

Now we will eliminate these patterns, mostly by considering where elements must go. (For convenience, here an n-line refers to a block containing n elements of X_2.) For patterns A and B, where $x \leq 2$, there is at least one X_1 element with at least 4 incidences on the last 11 blocks, but then the average number of pairs with X_2 elements in these blocks is, at most, $19/4$ so we need a block with at most 4 X_2 elements in the last 11 blocks. But Table 1 shows that this is a contradiction.

Now suppose there is an element, p, of frequency $6 + y$ in X_2. Let us consider the X_2 vs. p pairs. Define $\delta(p)$ to be the number of X_1 vs. p pairs minus the number of elements in X_1. Now, p occurs with $(6+y)12 = 72 + y$ other elements of which 22 are in X_2^c. This leaves at least $72+y-22 = 50+y - \delta(p)$ X_2 vs. p pairs. By examining Table 1 we see that this number is at most 42 (for case T). This is a contradiction. So there are 5 elements of frequency 5 in X_2; let p be one of them. Then p must occur in at least 60-22=38 X_2 vs. p pairs. Now, except for patterns F and L, we must have

Table 1: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_1 vs X_2
–	0	None	N/A	N/A
A	1	$10^2 9^2; 7^1 6^2 5^4$	253	0
B	2	$10^2 9^2; 7^2 6^1 5^4$	259	0
C	3	$10^2 9^2; 7^2 6^2 5^3$	264	2
D	3	$10^2 9^2; 7^2 6^3 5^1 4^1$	265	0
E	3	$10^2 9^2; 7^3 5^4$	265	0
F	4	$10^2 9^2; 7^2 6^3 5^2$	269	4
G	4	$10^2 9^2; 7^2 6^4 4^1$	270	2
H	4	$10^2 9^2; 7^3 6^1 5^3$	270	2
I	4	$10^2 9^2; 7^3 6^2 5^1 4^1$	271	0
J	4	$10^2 9^2; 8^1 7^1 6^2 5^3$	271	0
K	4	$11^1 10^1 9^1 8^1; 7^2 6^3 5^2$	271	0
L	5	$10^2 9^2; 7^2 6^4 5^1$	274	6
M	5	$10^2 9^2; 7^3 6^2 5^2$	275	4
N	5	$10^2 9^2; 7^4 5^3$	276	2
O	5	$10^2 9^2; 7^3 6^3 4^1$	276	2
P	5	$11^1 10^1 9^1 8^1; 7^2 6^4 5^1$	276	2
Q	5	$10^2 9^2; 7^4 6^1 5^1 4^1$	277	0
R	5	$10^2 9^2; 8^1 7^2 6^1 5^3$	277	0
S	5	$10^2 9^2; 8^1 7^1 6^4 4^1$	277	0
T	5	$11^1 10^1 9^1 8^1; 7^3 6^2 5^2$	277	0

at least one frequency 5 element with $\delta(p) = 0$, and so this element occurs with less than 38 X_2 elements which is a contradiction. Finally, in patterns F and L, we must have at least one element of frequency 5 with $\delta(p) \leq 1$, and here again these elements occur with less than 38 X_2 elements. So this establishes the result for $r \geq 3$.

Now we look at $r = 2$. Evaluating the difference at $r = 2$, we have:

$$(4m^2 + 48m + 4mx - 52 - 11x + x^2)/14.$$

This evaluates as $(x^2 - 7x)/14$ for $m = 1$ and $(x^2 - 3x + 60)/14$ for $m = 2$, and the monotonicity over m shows we have a positive difference for $m \geq 2$, i.e., that we must generate more pairs than we need if $m \geq 2$.

If we use the stricter inequality in Lemma 3.3 with $m = 1$, then we find that the minimum number of X_2 vs. X_2 pairs is generated with a distribution of $7^4 4^{7-x} 5^x$ of the X_2 elements over the last 11 blocks, and this number of pairs is $126 + 4x$ which is exactly the maximum needed, so we cannot vary the X_2 distribution. But now an X_2 element with frequency 4 must meet at least $6 + 6 + 3 + 3 = 18$ other elements of X_2 and only 17 are allowed. So we have established the lemma for $r = 2$ if $|B_i \cap B_j| \geq 4$. The remainder of the proof for the case $|B_i \cap B_j| = 3$ when $r = 2$ will be given in a later section. ■

At this point, we have to show that the covering design actually came from the projective plane by duplicating certain of the elements wherever they occur in the design. The major difficulty is the elements of frequency more than 4. So we will now define an *i-block* to be a block that contains exactly i elements of frequency greater than 4. The first step is to show there are no 0-blocks.

Lemma 4.4 *There does not exist a* 0*-block in a* $(13r+2, 4r+1, 2)$*-Covering Design on* 13 *blocks with* $r \geq 2$*.*

Proof: Assume there is a 0-block, B_0, in a $(13r + 2, 4r + 1, 2)$-Covering Design on 13 blocks. Assume that the largest intersection, t, satisfies $t \geq r + 5$. Let $t = t_{i,j} = r + 5 + m$, $m \geq 0$. At most 5 of the elements in the intersection of B_i and B_j, are of frequency greater than 4. Let 1 be an element of frequency 4 occurring in B_i, B_j, B_1, B_2. So by Lemma 4.1 and Lemma 4.3, B_1 and B_2 contain at least $r + m$ elements from the intersection of B_i and B_j. Since there are $7r - 3 - m$ distinct elements in $B_i \cup B_j$, there are $(13r + 2) - (7r - 3 - m) = 6r + 5 + m$ elements in $X_2 = X \setminus (B_i \cup B_j)$. These X_2 elements must also occur in B_1 or B_2. Thus there are $2(r + m) + 6r + 5 + m = 8r + 5 + m > 8r + 2$ elements in B_1 and B_2. This is a contradiction so $t \leq r + 4$. Consider B_0, the 0-block. It must intersect one block in at least $\left\lceil \frac{3(4r+1)}{12} \right\rceil = r + 1$. Say the intersection is $r + 1 + m$, with $m \geq 0$. By Lemma 4.1 and Lemma 4.3, the frequency 4 elements all occur together or not at all. So we have four blocks all sharing the same $r + 1 + m$ elements, but then element 1 has an excess of $3(r + m) > 3r - 1$ which is a contradiction. ■

There are at most 25 occurrences of elements in the covering design where the element has frequency greater than 4. This is an average of 1 and 12/13 such occurrences per block. Since 0-blocks have been ruled out we always get a 1-block. We record this in the next lemma.

Lemma 4.5 *In every* $(13r+2, 4r+1, 2)$*-Covering Design on* 13 *blocks with* $r \geq 2$*, there always exist a* 1*-block.*

We need the following definitions. Let $t'_{i,j}$ be the the number of elements of frequency 4 in the intersection of blocks B_i and B_j. Let $u \in X$ be an element of frequency 4 and define $SF(u)$ to be the set of elements that are contained in exactly the same set of blocks that contain u.

Lemma 4.6 *In a* $(13r + 2, 4r + 1, 2)$*-Covering Design on* 13 *blocks with* $r \geq 2$*, it must be that* $t \leq r + 3$*,* $t'_{i,j} \leq r$ *and that if* $t'_{i,j} = r$*, then* $B_i \cap B_j = SF(u)$ *where* $u \in B_i \cap B_j$*.*

Proof: Assume that $t = t_{i,j} = r + 4 + m$, $m \geq 0$, and let B_1 and B_2 be the other two blocks that contain u. Since there are at most 5 elements in the covering with frequency greater than 4, $B_i \cap B_j$ contains at least $r-1+m$ elements of frequency 4. Since Lemma 4.1 and Lemma 4.3 apply, we know that both B_1 and B_2 contain those $r-1+m$ frequency 4 elements. Let $X_2 = X \setminus (B_i \cup B_j)$. So $|X_2| = v - 2k + t = 6r + 4 + m$ and these elements must occur at least once in B_1 and B_2. Then B_1 and B_2 contain at least $6r + 4 + m + 2(r - 1 + m) = 8r + 2 + 3m$ elements. But since the sum of their block sizes is $8r + 2$, this implies that $m = 0$.

Now consider $t = t_{i,j} = r + 4$, and that the total frequency of the X_2 elements is $4|X_2| + x$. The X_2 pairs are all frequency 4 and we know for them that $\epsilon = 3r - 1$ so there are $(6r + 4)(3r - 1 + 6r + 3)/2 + 4rx/2 = 27r^2 + 24r + 4 + 2rx$ X_2 vs. X_2 pairs. But there are at least $(3r+2)(3r+1) = 9r^2 + 9r + 2$ X_2 pairs in blocks B_1 and B_2. Further, the X_2 elements, to minimize the number of X_2 pairs, occur in $3 + x$ blocks containing $2r + 2$ elements and $6 - x$ blocks containing $2r + 1$ elements. So there are at least $27r^2 + 24r + 5 + 2rx + x$ X_2 vs. X_2 pairs which is $1 + x$ too many. So $t \le r + 3$.

Now suppose that $t'_{i,j} = r + 1 + n$ with $n \ge 0$. Then $t - t_{i,j} \le t - t'_{i,j} \le 2 - n \le r$, so by Lemmas 4.1 and 4.3 the element u has at least $3(t'_{i,j} - 1) = 3(r + n)$ excess pairs, but only $3r - 1$ are allowed. Hence $t'_{i,j} \le r$. Also, if $t_{i,j} \ge r + 1$, then Lemmas 4.1 and 4.3 imply that all the frequency 4 elements in the intersection always occur together in the covering. So we just have to prove the final result for $t_{i,j} = t'_{i,j} = r$. So there are $2k - t_{i,j} = 7r + 2$ distinct elements in $B_i \cup B_j$ and $(13r+2)-(7r+2) = 6r$ elements in $X_2 = X \setminus (B_i \cup B_j)$. Since B_i and B_j are not 0-blocks they each contain at least one element of frequency greater than 4, and since $t_{i,j} = t'_{i,j}$ these elements are distinct, so of the $6r$ elements in X_2, at least $6r - 3$ of them have frequency 4 and at most 3 have frequency greater than 4. Assuming the opposite of the lemma, let one element of frequency 4 in the intersection of blocks B_i and B_j also occur in B_1 and B_2 and another element of frequency 4 also occur in B_3 and B_4. (Note that Lemma 4.1 still does apply, and $t - t'_{i,j} \le 3$, so this is the only alternative when $r \ge 3$.) Then, one of B_1 or B_2, say B_1, must contain at least $3r - 1$ elements of frequency 4 from X_2. Call this set of elements Y. Then one of B_3 or B_4, say B_3, must contain at least $\lceil (3r - 1)/2 \rceil > r$ elements from Y. Then B_1 and B_3 contain at least $r + 1$ elements of frequency 4 in their intersection but this contradicts the previous part of the proof.

In the $r = 2$ case, we must also consider the case that $t'_{i,j} = 2$ when $t = 5$, so Lemma 4.1 is not applicable. So now we must exclude the case that element 1 appears in blocks B_i, B_j, B_1 and B_2, and element 2 appears in blocks B_i, B_j, B_1 and B_3. Say B_i contains elements 3 through 9, B_j contains elements 10 through 16, B_1 contains elements 17 through 23, and B_2 and B_3 each contains elements 24 through 28. Since B_i, B_j and B_3 are not 0-blocks, each must contain at least one element of frequency 5 or more, so $|B_2 \cap B_3| = 5$, and $B_2 \cap B_3$ contains at least a triple of frequency 4 elements, so now Lemmas 4.1 and 4.3 cause this triple to be repeated in four blocks, generating too many pairs within the triple. ■

Lemma 4.7 (Todorov [6]) *Let B_0 be a 1-block in a $(13r+2, 4r+1, 2)$-Covering Design with $r \geq 2$. Then $B_0 = \cup_{i=0}^{3} SF(\alpha_i) \cup \beta$ where β has frequency greater than 4 and $|SF(\alpha_i)| = r$. Moreover if B_j, with $j > 0$, is another block in the covering design, then $B_0 \cap B_j = SF(\alpha_i)$ for some i or $B_0 \cap B_j = SF(\alpha_i) \cup \beta$.*

Proof: Let $\beta \in B_0$, and let β have frequency $4 + x$, with $0 < x \leq 5$. Since $t'_{i,j} \leq r$ and B_0 is a 1-block, the largest intersection with B_0 is $r + 1$. Moreover, since β has frequency $4 + x$, B_0 has at most $3 + x$ intersections of size $r + 1$, all including the element β. Since there are $12r + 3 + x$ other occurrences of the elements of B_0 somewhere in the 12 other blocks of the design, B_0 has $3 + x$ intersections of size $r + 1$ and $12 - x - 3$ intersections of size r. The intersections of size $r + 1$ must consist of β and r elements of frequency 4. This uses up β in the design and all other intersections with B_0 consist of r elements of frequency 4. Applying the previous lemma then gives our result. ∎

From this point on, we treat only $r \geq 3$ and defer the completion of the case $r = 2$ until the next sections.

Lemma 4.8 (Todorov [6]) *In a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks with $r \geq 3$, then $t \leq r+2$ where t is the largest intersection of two blocks in the covering.*

Proof: Since $t \leq r+3$ by Lemma 4.6, the only case to rule out is $t = r+3$. So let us assume that $|B_i \cap B_j| = r + 3$. Let B_i contain the elements from 1 to $4r + 1$ and B_j contain 1 to $r + 3$ and $4r + 2$ to $7r - 1$. Let $X_2 = X \setminus (B_i \cup B_j)$. Then $|X_2| = (13r + 2) - (7r - 1) = 6r + 3$. Let element 1 be of frequency 4 and let it occur in B_1 and B_2. The elements of X_2 must occur at least once in B_1 or B_2, so at least one of the blocks, say B_1, contains at least $3r + 2$ X_2 elements. So B_1 and B_2 contain at most $r - 1$ elements (including the element 1) from the intersection $B_i \cap B_j$, and say these elements are $1, 2, \ldots, r - 1$. Now if element r had frequency 4, then it would occur in B_1 and B_2 by Lemmas 4.1 and 4.3, so element r must have frequency greater than 4, and, since B_1 has only $r - 1$ elements from B_i, $r \notin B_i$. Similarly, $r + 1$, $r + 2$ and $r + 3$ must be elements of frequency greater than 4 and they do not occur in B_1. So B_1 must be a 1-block with one of the elements from $1, 2, \ldots, r - 1$ being the element of frequency greater than 4. But $|B_i \cap B_1| = r - 1$ and B_1 is a 1-block, violating Lemma 4.7. ∎

Lemma 4.9 (Todorov [6]) *Consider a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks with $r \geq 4$. Let $X_1 = \bigcup_{i=1}^{m} SF(\alpha_i)$. If the elements of frequency greater than 4 are removed and $SF(\alpha_i)$, $i = 1, 2, \ldots, m$ is replaced by α_i, a Steiner System $S(13, 4, 2)$ is obtained. If $r = 3$, and a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks exists, then there also exists a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks which satisfies the above property of transformability to a $S(13, 4, 2)$.*

Proof: Consider any block B_0. It must contain an element of frequency 4, say α. Then let $SF(\alpha)$ occur in blocks B_1, B_2, B_3 and B_4. Now $|SF(\alpha)| \leq r$ so B_1 contains other elements of frequency 4, call one of them β, $\beta \notin SF(\alpha)$. $SF(\beta)$ can not occur in B_2, B_3 or B_4, as otherwise it would be a subset of SF(α). Since Lemma 4.1 applies, $SF(\beta)$ can not occur in just 3 of the blocks of B_1, B_2, B_3 and B_4. But $SF(\beta)$ could occur in two of the blocks of B_1, B_2, B_3 and B_4. This could happen if $|SF(\alpha) \cup SF(\beta)| \leq r$ and then Lemma 4.3 does not apply. So let $SF(\beta)$ occur in blocks B_1, B_2, B_5 and B_6 and let $|SF(\alpha) \cup SF(\beta)| = r - x$, where $x \geq 0$. Then the maximum number of distinct elements in $B_1 \cup B_2$ is $7r + 2 + x$. This means that the minimum number of X_2 elements is $13r + 2 - 7r + 2 + x = 6r + x$, of which at least $6r - 5 + x$ are frequency 4 elements. They must all occur in B_3 or in B_4, so one of them, say B_3 contains at least $\lceil(6r - 5 + x)/2)\rceil$. Then one of B_5 or B_6 must intersect B_3 in at least $\lceil(\lceil(6r - 5 + x)/2)\rceil/2\rceil$ elements. But this is bigger than r for $r \geq 3$.

So let β occur in blocks B_1, B_5, B_6 and B_7. Again $|SF(\beta)| \leq r$, so B_1 contains an element, γ, of frequency 4, $\gamma \notin SF(\alpha) \cup SF(\beta)$. Arguing as before, let $SF(\gamma)$ occur in blocks B_1, B_8, B_9 and B_{10}. If $r > 4$, or $r = 4$ and B_1 is not a 5-block, or $r = 4$ and B_1 is neither a 4-block nor a 5-block, then B_1 contains another frequency 4 element, δ, where $\delta \notin SF(\alpha) \cup SF(\beta) \cup SF(\gamma)$. As before, $SF(\delta)$ can not occur in blocks $B_1, \ldots, B_{10}$ so let it occur in blocks B_{11}, B_{12} and B_{13}. Since there are no more blocks, B_1 can have no more elements of frequency 4. Repeat this process for all the other blocks. Now, if the elements of frequency 5 or more are deleted and the $SF(\alpha_i)$, $i - 1, \ldots, m$ are replaced by α_i throughout the design, we see that in the transformed design, each element occurs 4 times, each pair occurs once, and there are 13 blocks of size 4. Clearly, this an $S(13, 4, 2)$.

However, if $r \leq 4$, then some $SF(\delta)$s may not exist. This could happen if there are 5-blocks and $r \leq 4$, or there are 4-blocks and $r \leq 3$. Consider the transformed designs in these cases. Since there are at most 25 incidences of elements with frequency greater than 4, and there are no 0-blocks, there are at most three 5-blocks or at most four 4-blocks. Now if $r = 4$, then we get at least $(4 \cdot 10 + 3 \cdot 3)/4$ α_is, i.e., at least 13 α_is, so if $r = 4$ we get our $S(13, 4, 2)$.

However, when $r = 3$, then we get at least $(4 \cdot 9 + 3 \cdot 4)/4$ α_is, i.e., at least 12 α_is. Now these α_is each have frequency 4, each pair occurs once in some block, and there are either 3 or 4 α_is per transformed block, so if we don't have 13 α_is, then what we do have is an $S(13, 4, 2)$ with, say, α_{13} deleted. If there are 12 α_is, then each must occur with 3 others three times, and 2 others once, so the 4 transformed blocks with 3 α_is per transformed block contain all the 12 α_is once, and these are the blocks from which the point of the $S(13, 4, 2)$ was deleted. Now for this situation to arise, we must have four 4-blocks in the original design that transform to the blocks containing the puncture point of the punctured $S(13, 4, 2)$. There are also nine 1-blocks in this case. Since the four 4–blocks must contain a common element, we can replace this common element by an element, α_{13}, of frequency 4, then replace all the occurrences of α_{13} in 1-blocks by occurrences of some other element of frequency greater than 4. We have seen that α_{13} meets all the elements of frequency 4, and it must also meet all the elements with frequency greater than 4, since these only met in these four 4-blocks. ■

We now need a property of the Steiner System $S(13,4,2)$, i.e., $\mathrm{PG}(2,3)$, which can be verified by inspection. Alternatively, viewing the dual design, it is just the special case $p = 3$ of Blokhuis' result [3] that the minimum blocking set in $\mathrm{PG}(2,p)$ has size $3(p+1)/2$ where p is an odd prime.

Lemma 4.10 (Todorov [6]) *In a Steiner System $S(13,4,2)$, if a set of blocks contains all elements but not every occurrence of an element, then the number of blocks is at least* 6*.*

We are now ready for Todorov's Theorem.

Theorem 4.11 (Todorov [6]) *If $r \geq 3$, then $C(13r+2, 4r+1, 2) > 13$.*

Proof: Consider a $(13r+2, 4r+1, 2)$-Covering Design on 13 blocks with $r \leq 3$. (In the case that $r = 3$, we will transform any punctured Steiner Covering Design to the full Steiner Covering Design, then work with the full one. We will not formally define these terms, but the meaning should be clear from the proof of Lemma 4.9. By showing the full Steiner Covering Design cannot exist, we also show that the punctured Steiner Covering Design cannot exist either.)

If, in our covering design, x occurs with every occurrence of an element of frequency 4, then delete the occurrences of x in the blocks that do not contain that element of frequency 4. Note that x still occurs with every element because the element of frequency 4 occurs with every element, so this new design on $13r+2$ elements still has every pair occurring but some blocks may be shorter than $4r+1$. Any leftover elements of frequency greater than 4 must occur at least 6 times by Lemma 4.10. The elements in this design occur either 4 times or at least 6 times. Since there are at most 2 elements of frequency 6 or more, there must be at least $13r$ elements of frequency 4. We know from Lemma 4.9 that $|SF(\alpha_i)| \leq r$ for $i = 1, 2, \ldots, 13$. So $|SF(\alpha_i)| = r$ for $i = 1, 2, \ldots, 13$ and the number of elements of frequency 6 or more is exactly 2. But this leaves room for only one element of frequency 6 or more in each block. So no block can contain the pair of elements of frequency 6 or more. ■

5 A (28, 9, 2)-Covering Design

The objective of this section is to establish Lemma 4.3 for the case $r = 2$ which we state as Lemma 5.1. We begin by noting that Lemma 4.1 holds for $r = 2$.

Lemma 5.1 *In a* $(28, 9, 2)$*-Covering Design on* 13 *blocks, assume that two blocks,* B_i *and* B_j*, intersect in* $t_{i,j} \geq 3$ *elements. Then any two elements of frequency* 4 *in the intersection must occur together again.*

Proof: In the proof of Lemma 4.3, we established the result for $t_{i,j} > 3$, so we only need to show the result for $t_{i,j} = 3$. We will need to deal with a large number of cases, and we start by limiting this number. Assume that elements 1 and 2 have frequency 4 and occur in the intersection of B_i and B_j. Further assume that they do not occur together again in any other block. Then this is the situation:
$B_i : \{1, 2, 3, 4, \ldots, 9\}$
$B_j : \{1, 2, 3, 10, \ldots, 15\}$
$B_1 : \{1, \ldots\}$ $B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$ $B_4 : \{2, \ldots\}$
For the remainder of the proof, it is important to note that an element of frequency $(4 + x)$ occurs in $(5 + 8x)$ excess pairs. We now look at the possibility that some element(s) of X_2 in B_1 or B_3 are repeated in B_2 or B_4 respectively.

We can eliminate the possibility of two repeated elements. Suppose $X_2 = (B_i \cup B_j)^c$, and that the total number of incidences of X_2 elements is $52 + x$. In this case the total number of incidences of X_2 elements in blocks B_1–B_4 is 28, and the smallest number of pairs amongst the X_2 points is $114 + 3x$ for $x \leq 4$ (with $7^4 3^{4-x} 4^{3+x}$) and $110 + 4x$ for $x \geq 4$ (with $7^4 4^{11-x} 5^{x-4}$). The maximum number of pairs allowed is $110 + 4x$, so we must have $x \geq 4$, but then we are allowed at most one excess X_2 vs. X_2^c pair, and we have an extra pair with the first repeated element and element 1, and an extra pair with the second repeated element and element 2. Note also that, since we had equality of generated and allowed pairs, having more than 28 incidences of X_2 elements in blocks B_1–B_4 will clearly generate too many pairs.

Next we note that, by Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$, if a pair of frequency 4 elements occurs in any intersection of size 4 or more, then that pair occurs in the same four blocks. So no triple of frequency 4 elements can occur in an intersection of size 4 or more as then the triple would occur in the same four blocks, so each point would have an excess of 6 pairs in these four blocks, but only 5 are allowed for any element of frequency 4.

Now let A, B, C and D be disjoint sets and θ be an element of $X_2 \cap (B_1 \cap B_2)$. Let $X_2 \cap B_1 = A \cup B$, $X_2 \cap B_2 = C \cup D \cup \{\theta\}$, $X_2 \cap B_3 = A \cup C$ and $X_2 \cap B_4 = B \cup D$. It is also convenient to consider $\{\theta\} = \emptyset$ here.

We look at the various patterns of X_2 on B_1–B_4, and note in parentheses the number of X_2 vs. X_2 pairs generated by the pattern on these four blocks. We determine, by inspection, the minimum number of elements of frequency 5 or more that is required, and give an example of a partition that yields this minimum (when $\theta = a \in A$ some of these partitions require that a be an element of frequency 5 or more). For $7, 6, 7, 6$ (72): the minimum number of elements of frequency 5 or more is 2 and occurs when $|A| = 4$; for $7, 6, 8, 5$

(74) and $8, 5, 8, 5$ (76): the minimum number of elements of frequency 5 or more is 3 and occurs when $|A| = 5$; for $7, 7, 7, 6$ (78): the minimum number of elements of frequency 5 or more is 3 and occurs when $|A| = 4$ and $\theta = a \in A$; for $7, 7, 8, 5$ (80): the minimum number of elements of frequency 5 or more is 4 and occurs when $|A| = 5$ and $\theta = a \in A$; for $8, 6, 7, 6$ (79): the minimum number of elements of frequency 5 or more is 3 and occurs when $|A| = 5$ and $\theta = a \in A$; for $8, 6, 8, 5$ (81): the minimum number of elements of frequency 5 or more is 3 and occurs when $|A| = 5$ and $\theta = b \in B$. If there is no repeated element, then the most equitable distribution on the last 7 blocks (assuming $x \geq 2$) is $4^{9-x}5^{x-2}$ and generates $34 + 4x$ pairs; if there is a repeated element, then the most equitable distribution on the last 7 blocks (assuming $x \geq 3$) is $4^{10-x}5^{x-3}$ and generates $30 + 4x$ pairs. Comparison with the maximum allowed number of $110 + 4x$ eliminates the pattern starter $8, 6, 8, 5$. Note that the repeated element can meet at most 6 elements of X_2^c (including 1 and 2) on blocks B_1–B_4, and so needs at least two more blocks to meet the other 9 elements, and so the repeated element has a frequency 5 or more. Also note that the number of elements of frequency 5 or more is only a lower bound for x.

We enumerate the possible patterns with a repeated point in Table 2.

In the last column of Table 2, we list m_1, m_2, the number of excess pairs a frequency 5 element must have with X_2^c. We list the value m_1 for the repeated point first. The value is based purely on block sizes, and indeed the repeated point must have an extra pair with element 1. Now (except for case 5X), the value $m_2 \cdot (x - 1) + \max(m_1, 1)$ exceeds the allowable excess of X_2 vs. X_2^c pairs, so we cannot have a pattern with x frequency 5 elements, and since the fourth largest block in the last seven is a 4- or 5-line, changing two frequency 5 elements to a frequency 6 element never improves things, and changing three frequency 5 elements to a frequency 7 is even worse. So now we just need to look at pattern 5X with five frequency 5 elements: the only way we can place the non-repeated elements is in the blocks $8^1 7^1 5^3$, and we have at least 8 frequency 4 elements in X_2^c that do not appear in blocks B_1–B_4, and one of these must appear in the 3-line, and so must appear in two 5-lines, but the 5-lines can contain at most 7 distinct elements of X_2, and at least 10 are needed now. So we have no repeated point from X_2 in B_1–B_4.

So if the the total frequency of X_2 is $52 + x$, now we know that we need only consider $x \geq 2$. The last column of Table 3 lists m, the minimum excess of pairs with X_2^c that an X_2 element of frequency 5 must have. Since replacing two elements of frequency 5 by a single element of frequency 6 must increase this value by $4 - m$ if the fourth largest block in the last 7 is a 5-line, and $5 - m$ if it's a 4-line, and the largest value of m is 4, so if we exclude the pattern assuming x frequency 5 elements, we couldn't rescue the pattern by using other frequencies. Note that m is computed based solely on the block sizes, so if $x \cdot m$ exceeds the maximum allowable excess for X_2 vs. X_2^c, then the pattern fails, and we list these excluded patterns in Table 3.

Table 2: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
3P	3	$7^3 6^1; 4^7$	120	5	1, 4
3Q	3	$7^3 6^1; 3^1 4^5 5^1$	121	3	0, 3
3R	3	$7^3 6^1; 3^2 4^3 5^2$	122	1	−1, 2
3S	3	$8^1 7^1 6^2; 4^7$	121	3	1, 3
3T	3	$8^1 7^1 6^2; 3^1 4^5 5^1$	122	1	0, 2
4R	4	$7^3 6^1; 4^6 5^1$	124	5	0, 3
4S	4	$7^3 6^1; 3^1 4^4 5^2$	125	3	−1, 2
4T	4	$7^3 6^1; 3^2 4^2 5^3$	126	1	−1, 1
4U	4	$7^3 6^1; 3^1 4^5 6^1$	126	1	−1, 2
4V	4	$8^1 7^1 6^2; 4^6 5^1$	125	3	0, 2
4W	4	$8^1 7^1 6^2; 3^1 4^4 5^2$	126	1	−1, 1
4X	4	$8^1 7^2 5^1; 4^6 5^1$	126	1	−1, 2
5R	5	$7^3 6^1; 4^5 5^2$	128	5	−1, 2
5S	5	$7^3 6^1; 3^1 4^3 5^3$	129	3	−1, 1
5T	5	$7^3 6^1; 4^6 6^1$	129	3	−1, 2
5U	5	$7^3 6^1; 3^2 4^1 5^4$	130	1	−1, 1
5V	5	$7^3 6^1; 3^1 4^4 5^1 6^1$	130	1	−2, 1
5W	5	$8^1 7^1 6^2; 4^5 5^2$	129	3	−1, 1
5X	5	$8^1 7^1 6^2; 3^1 4^3 5^3$	130	1	−1, 0
5Y	5	$8^1 7^1 6^2; 4^6 6^1$	130	1	−1, 1
5Z	5	$8^1 7^2 5^1; 4^5 5^2$	130	1	−2, 1

Now consider that we have at least $15-(5-x)$ elements of frequency 4 in X_2^c. Moreover assume element 3 in $B_i \cap B_j$ has frequency 4. Then, by Lemma 4.1 it cannot lie in just one block of B_1, B_2, nor just one block of B_3, B_4. We now show it must lie in two blocks of one of these pairs. If we suppose not, then it lies in B_5, B_6, say, and B_5, B_6 together must contain all 13 elements of X_2, but there is no pattern for which this is possible. (The patterns yet to be excluded are listed in Tables 4–7.) Since $x = 5$ entails element 3 has frequency 4, element 3 must occur in either B_1 and B_2, or B_3 and B_4 which implies a block with 10 elements (2 of 1–3 and 8 of X_2) in pattern 5Q. This eliminates pattern 5Q. So, if element 3 has frequency 4, then elements 1, 2 and 3 occur a total of 6 times in B_1–B_4 and there are at most 4 other elements of X_2^c in B_1–B_4. So there are at least $(12-4)-(5-x) = 3+x$ elements of X_2^c of frequency 4 with one incidence in the first 6 blocks. Alternatively, if element 3 has frequency 5 and does not occur in blocks B_1–B_4, then elements 1 and 2 occur four times in B_1–B_4, and there are at most 6 other elements of X_2^c in B_1–B_4, so there are at least $(12-6)-(5-x-1) = 2+x$ elements of frequency 4 with one incidence in the first 6 blocks. These $2+x$ elements along with element 3

Table 3: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
2D	2	$7^2 6^2; 2^1 4^4 5^2$	117	3	2
2F	2	$7^2 6^2; 3^2 4^4 6^1$	117	3	2
2G	2	$7^2 6^2; 2^1 3^1 4^2 5^3$	118	1	1
2H	2	$7^2 6^2; 2^1 4^5 6^1$	118	1	2
2I	2	$7^2 6^2; 3^3 4^2 5^1 6^1$	118	1	1
3D	3	$7^2 6^2; 3^1 4^5 6^1$	120	5	2
3H	3	$7^2 6^2; 2^1 3^1 4^1 5^4$	122	1	1
3I	3	$7^2 6^2; 2^1 4^4 5^1 6^1$	122	1	1
3K	3	$8^1 7^1 6^1 5^1; 4^6 5^1$	120	5	2
3N	3	$8^1 7^1 6^1 5^1; 3^1 4^5 6^1$	122	1	1
3O	3	$8^2 5^2; 4^6 5^1$	122	1	1
4C	4	$7^2 6^2; 4^6 6^1$	123	7	2
4F	4	$7^2 6^2; 2^1 4^2 5^4$	125	3	1
4H	4	$7^2 6^2; 2^1 3^1 5^5$	126	1	1
4N	4	$8^1 7^1 6^1 5^1; 4^6 6^1$	125	3	1
5F	5	$7^2 6^2; 2^1 4^1 5^5$	129	3	1

have to meet at least 13 elements of X_2 with three incidences on the last 7 blocks. If element 3 has frequency 5 and does occur in blocks B_1–B_4 or if element 3 has frequency 6 or more, then there are at least $x+3$ elements of frequency 4 with one incidence in the first 6 blocks. So in every case we have $x+3$ elements that must occur 3 times in the last 7 blocks with all the X_2 elements. These criteria eliminate the patterns in Table 4.

We next eliminate the remaining patterns containing an 8-line which are all listed in Table 5.

For pattern 5O: let the 8-line intersect the 7-line in A and the 6-line in B. Also let the 5-line intersect the 7-line in C and the 6-line in D. We have 5 frequency 5 elements in X_2. Using the usual arguments, we get $|A| \leq 5$ and hence $|B| + |C| \geq 5$. In this case, every element in $B \cup C$ must contribute an even number of X_2 vs. X_2^c pairs, and so have an excess of one extra pair with X_2^c. Thus we have at least 5 extra pairs with X_2^c, and only 1 is allowed.

For patterns 4L and 5L, a frequency 4 element in the 8-line (which must exist) must meet at least $8+6+4+4-4 = 18$ X_2 elements, which is one too many.

In pattern 4Q, there are at least 4 elements of frequency 4 in the 8-line containing element 1. To avoid the contradiction in the previous paragraph, these elements must occur in the 5-line containing element 2. But the part of Lemma 4.3 already proved for $r = 2$ implies this quadruple will occur together on the same four lines, generating too many repeated pairs

Table 4: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
2A	2	$7^2 6^2; 4^7$	114	9	4
2B	2	$7^2 6^2; 3^1 4^5 5^1$	115	7	3
3A	3	$7^2 6^2; 4^6 5^1$	118	9	3
4K	4	$7^2 6^2; 3^2 4^3 6^2$	126	1	0
5C	5	$7^2 6^2; 4^5 5^1 6^1$	127	7	1
5H	5	$7^2 6^2; 3^1 4^4 6^2$	129	3	0
5K	5	$7^2 6^2; 3^1 4^4 5^1 7^1$	130	1	0
5N	5	$8^1 7^1 6^1 5^1; 4^5 5^1 6^1$	129	3	0
5Q	5	$8^2 5^2; 4^4 5^3$	130	1	(-1)

amongst the quadruple.

For patterns 3L, 4M and 4P, a frequency 4 element in the 8-line must meet at least $8+6+4+4-4 = 18$ X_2 elements (which is one too many) unless it lies in the 3-line, but there are only 3 elements for which this can happen, and we have at least $8-x > 3$ frequency 4 elements in the 8-line. Contradiction.

For patterns 5M and 5P, the preceding argument implies that we must have five frequency 5 elements in the 8-line. The three frequency 4 elements in the 8-line must lie in the unique 8- 6- and 3-lines and, by Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$, in the same 4-line and so they generate too many repeated pairs amongst the triple.

For pattern 3M, there is at least one element of frequency 5 or more in X_2^c with 4 incidences on the last 11 blocks, and so this element must meet at least $3+3+4+4$ elements of X_2. Having 5 incidences on the last 11 blocks generates too many X_2 vs. X_2^c pairs, but we could still have an element of frequency 6 in $B_i \cap B_j$. This $3+3+4+4$ is a unique pattern, and uses up all the allowed X_2 vs. X_2^c excess. Now consider the X_2 points in the 8-line. There must be at least 5 of frequency 4, and these must have a pattern of $8^1 6^1 3^1 4^1$, $8^1 7^1 3^2$ or $8^1 6^1 3^2$ to avoid too many X_2 pairs, but each of these will have two pairs with the frequency 5 element in X_2^c, so we must have at least 5 extra X_2 vs. X_2^c pairs, and only 1 is allowed.

In pattern 4O, a frequency 5 element in X_2^c with 4 incidences on the last 11 blocks must meet at least $3+3+4+5$ elements of X_2 which would exceed the allowed X_2 vs. X_2^c excess, so the frequency 5 element in X_2^c must be in $B_i \cap B_j$. Now look at the 5 X_2^c incidences on the 4-line. The only way to use these without causing excess pairs is with patterns 6-3-4 (at most 2 elements), and with 7-3-4 or 4-5-5 causing one excess pair, so using up all X_2^c incidences on 4-lines causes at least 3 excess X_2 vs. X_2^c pairs, and only 1 is allowed. This eliminates the last possible pattern in Table 5.

Table 5: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
3L	3	$8^17^16^15^1; 3^14^45^2$	121	3	1
3M	3	$8^17^16^15^1; 3^24^25^3$	122	1	0
4L	4	$8^17^16^15^1; 4^55^2$	124	5	1
4M	4	$8^17^16^15^1; 3^14^35^3$	125	3	0
4O	4	$8^17^16^15^1; 3^24^15^4$	126	1	0
4P	4	$8^17^16^15^1; 3^14^45^16^1$	126	1	0
4Q	4	$8^25^2; 4^55^2$	126	1	0
5L	5	$8^17^16^15^1; 4^45^3$	128	5	0
5M	5	$8^17^16^15^1; 3^14^25^4$	129	3	0
5O	5	$8^17^16^15^1; 3^25^5$	130	1	0
5P	5	$8^17^16^15^1; 3^14^35^26^1$	130	1	(-1)

It is convenient at this point to see if any of the elements in X_2 in the remaining viable patterns (listed in Tables 6 and 7) have frequency 6 or more. An element of frequency 9, 8 or 7 in X_2 has too many excess X_2 vs. X_2^c pairs all by itself. Similarly, a pair of frequency 6 elements also has too many excess X_2 vs. X_2^c pairs all by themselves. Finally, if X_2 has an element of frequency 6 and $x - 2$ elements of frequency 5, they will create too many excess X_2 vs. X_2^c pairs except for two possible patterns: 5A ($7^26^2; 4^45^3$) and 5E ($7^26^2; 3^14^35^26^1$), both with $x = 5$.

In pattern 5A: a frequency 4 element on the two 7-lines cannot meet all of X_2^c. Now the three elements of frequency 5 must be placed in the two 7-lines and the three 5-lines and the element of frequency 6 must also be placed in these blocks with an occurrence in a 4-line which uses up all the allowable X_2 vs. X_2^c excess pairs. So the 6-lines contain just frequency 4 elements from X_2. Exactly 3 of them can occur in the same two 6-lines. If more than 3, then the Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$ implies those 4 or more elements always occur together and hence their excess is too large. Now, consider the three frequency 4 elements in X_2 that occur in one of the 7-lines and one of the 6-lines. It must occur in a pattern of $7^16^14^2$ in order to occur with all the X_2^c elements. Since there are only four 4-lines, two of those elements must meet again and by Lemma 4.1, these elements occur together in all blocks. The remaining element cannot occur with the other two elements in the 4-line as again by Lemma 4.1 they would all 3 occur together in 4 blocks causing their excess to be too large. So the remaining element is disjoint from the first two elements in the 4-lines. Now this is true for the three elements of frequency 4 in X_2 that are in the other 7-line. There is only one way to fit them in so that the two sets of three elements meet in some block. This leaves three elements of frequency four that must occur in the pattern $6^24^15^1$. Now consider an element (necessarily of frequency 4) from $X_2^c \setminus (B_i \cap B_j)$ that occurs on a 6-line, but not on a 7-line. There is only one way in its two remaining blocks that this element can can meet all the X_2 elements. It must be on the two 6-lines and a 5-line as well as, say, B_i, but then it meets the 3 points of $B_i \cap B_j$ twice and the 3 common points of the 6-lines twice, but only 5 extra pairs are allowed.

The argument for pattern 5E starts the same as the 5A pattern argument. The elements of frequency more than 4 use up all the excess X_2 vs. X_2^c pairs so they occur in the pattern $7^2 5^2 6^1$ with an extra incidence on a 4-line for the element of frequency 6. The three elements of frequency 4 that lie on the first 7-line and the second 6-line must lie on a 3-line and a 5-line or else on two 4-lines. If we had two or more of these elements on the 3-line we would have these two or more elements on the same 5-line by Lemma 4.1. However, all 5-lines contain 4 elements of frequency more than 4, so we must have at most one element with a $7^1 6^1 3^1 5^1$ pattern and at least two elements with an identical $7^1 6^1 4^2$ pattern. The same argument applies to the three elements of frequency 4 that lie on the second 7-line and the first 6-line, and, since a 4-line only contains 4 X_2 elements, we must have one element with a $6^1 7^1 3^1 5^1$ pattern and two elements with an identical $6^1 7^1 4^2$ pattern. However the first element with the $7^1 6^1 3^1 5^1$ pattern cannot meet the elements in the second pair with the $6^1 7^1 4^2$ pattern.

So all elements of X_2 with a frequency more than 4 have frequency 5.

Suppose B_1 and B_3 are 7-lines and B_2 and B_4 are 6-lines. Let $A = X_2 \cap B_1 \cap B_3$, $B = X_2 \cap B_1 \cap B_4$, $C = X_2 \cap B_2 \cap B_3$ and $D = X_2 \cap B_2 \cap B_4$. Now, if any of A–D has size $n > 3$, then by Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$, that intersection must contain at least $n - 2$ elements of frequency 5 or we will have 4 blocks with a common triple of frequency 4 elements generating at least 6 extra pairs for one of those elements when only 5 are allowed. At this point, we see all the open cases (see Tables 6 and 7) must have $|A| = 4$, except possibly cases 5A, 5E and 5J. If we compute mx for Tables 6 and 7, where m is the minimum number of extra X_2 vs. X_2^c pairs generated by a frequency 5 element in X_2 (listed in the last column of the tables) and compare this with the actual number of pairs allowed, we see the difference is less than 4 (except cases 5A, 5E and 5J). Now m is always computed for an element in A, and the minimum will be increased by 1 for an element in $B \cup C$, and by 2 for an element in D. Now $|A| = 5$ implies $|D| = 4$, needing two frequency 5 elements in D, and $|A| = 3$ implies $|B| = |C| = 4$, needing two frequency 5 elements in B and in C, so our only option (except cases 5A, 5E and 5J) is to have $|A| = 4$. This also entails having at least two frequency 5 elements in A.

We now proceed to eliminate the remaining cases one by one. They are listed in Table 6 and Table 7.

For pattern 5D: we are allowed 5 excess X_2 vs. X_2^c pairs, and the frequency 5 pattern of $7^2 5^3$ yields one extra pair, so we must use this pattern for all frequency 5 elements. But this entails $|A| \geq 5$, contradicting $|A| = 4$.

For pattern 4J with its three 3-lines: we will show that there is an element of frequency 4 in X_2^c that does not meet all elements in X_2. The 3-lines allow for 18 incidences of X_2^c. The only element of X_2^c that could appear three times in the 3-line blocks is the frequency 5 element. There could be elements of frequency 4 in X_2^c that appear twice in the 3-lines but they would also have to occur with a 7-line. Allowing for elements 1 and 2, there are at most 2 of these. This leaves at least $18 - 3 - 2 \cdot 2 = 11$ occurrences for at most $15 - 3 - 2 = 10$ elements in X_2^c. Contradiction.

For pattern 3F: recall that $|A| = 4$, so $|A| + |D| = 7$, and we have 7 elements in $A \cup D$ contributing an even number of X_2 vs. X_2^c pairs from blocks B_1–B_4, and at least 4 of these have frequency 4. So they contribute an even total of X_2 vs. X_2^c pairs in the whole covering. However, $|X_2^c| = 15$ is odd, so we have at least 4 extra X_2 vs. X_2^c pairs, and only 3 are allowed.

For pattern 5I: $|A| = 4$, and we must have four frequency 5 elements with pattern $7^2 5^2 6^1$ and one with pattern $7^1 6^1 5^2 6^1$ (which uses up the excess X_2 vs. X_2^c pair). For elements of D we must use the pattern $6^2 4^1 5^1$ three times, and for the remaining 5 elements of $B \cup C$ we must use either the pattern $7^1 6^1 2^1 6^1$ or the pattern $7^1 6^1 4^2$. So we can only use 13 incidences on 5-lines, and there is no way to use the last two of these incidences.

For pattern 3E, we must give the three frequency 5 elements a pattern of $7^2 5^3$ which uses up all our excess X_2 vs. X_2^c pairs, so the frequency 4 patterns must all be exact, and the only possible pattern with a 2-line is $7^2 2^1 5^1$. However, we need to use two of these and so our 7-lines intersect in 5 elements, contradicting $|A| = 4$.

For pattern 3G, we must give the three frequency 5 elements a pattern of $7^2 4^1 5^1 6^1$ which uses up all our excess X_2 vs. X_2^c pairs, so the frequency 4 patterns must all be exact, and the only possible pattern using the remaining 3 incidences on the final 6-line is $6^2 3^1 6^1$. So by Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$, the 3-lines in the above frequency 4 pattern coincide, and we have a common triple of frequency 4 elements which generates too many pairs within the triple.

For pattern 5J, the basic patterns that give no excess X_2 vs. X_2^c pairs for the $X_2^c \setminus (B_i \cap B_j)$ elements are 1 of 7-3-3, 3 of 6-3-4 and 8 of the following: either 4-4-5 or 3-4-6. To get the excess of 1, a non-basic pattern must be used and none of them use both of the two 6-lines in the last 7 blocks. So the 6 incidences with the final two 6-lines mean that there are at least 2 elements of the type 4-4-5. But these elements can not coincide with the element with the first pattern so that element must have a non-basic pattern of 7-3-4 giving our excess X_2 vs. X_2^c pair. But now we have 10 incidences on the 3-lines instead of 12. Contradiction.

For patterns 4E and 5E: we can use at most x incidences of frequency 5 elements on the final 6-line but this still leaves some incidences on the 6-line which can only be used in the pattern $6^2 3^1 6^1$. Now, after allowing for the elements 1 and 2, we have 7 available X_2^c incidences on 6-lines and 6 on the 3-line, and 13 elements of $X_2^c \setminus \{1, 2\}$, so each of these elements must lie on exactly one 6-line or one 3-line. For pattern 4E: consider the 6

Table 6: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
3E	3	$7^2 6^2; 2^1 4^3 5^3$	121	3	1
3F	3	$7^2 6^2; 3^3 5^4$	121	3	1
3G	3	$7^2 6^2; 3^2 4^3 5^1 6^1$	121	3	1
4E	4	$7^2 6^2; 3^1 4^4 5^1 6^1$	124	5	1
4I	4	$7^2 6^2; 2^1 4^3 5^2 6^1$	126	1	0
4J	4	$7^2 6^2; 3^3 5^3 6^1$	126	1	0
5D	5	$7^2 6^2; 3^2 5^5$	128	5	1
5E	5	$7^2 6^2; 3^1 4^3 5^2 6^1$	128	5	0
5I	5	$7^2 6^2; 2^1 4^2 5^3 6^1$	130	1	0
5J	5	$7^2 6^2; 3^2 4^2 5^1 6^2$	130	1	(-1)

incidences on the 3-line. We can use one with the frequency 5 element in X_2^c and 7-line with a 4-line for two others. But now we only have 4-lines and one 5-line left, so we can only meet $3+4+5=12$ elements of X_2 with the final 3 incidences on the 3-line, not the 13 required. For pattern 5E: we must have a frequency 4 element in $B_i \cap B_j$, so it must lie on a 7-line and a 6-line. Now consider the remaining 6 incidences on the 6-lines: the best we can do is use two 4-lines with each 6-line, but each causes one excess X_2 vs. X_2^c pair, and only 5 are allowed.

For pattern 4I: we have $|A|=4$, and are allowed one excess X_2 vs. X_2^c pair. The only way to use the last 6-line is to use one incidence for each frequency 5 element and use each of the last two incidences with a 2-line. Since this prohibits the pattern $7^2 2^1 5^1$, we must have all four frequency 5 elements in A. Since we must use the pattern $7^2 5^2 6^1$ at least three times, every element in D must have an incidence on a 5-line or the final 6-line. So our possible patterns for X_2 are: 1) $7^2 5^2 6^1$ four times, $7^1 6^1 2^1 6^1$ once, $7^1 6^1 4^2$ five times, $6^2 2^2 6^1$ once and $6^2 4^1 5^1$ twice; or 2) $7^2 5^2 6^1$ three times, $7^2 4^1 5^1 6^1$ once, $7^1 6^1 2^1 6^1$ twice, $7^1 6^1 4^2$ four times and $6^2 4^1 5^1$ three times. In either case, we cannot have the pattern $7^1 6^1 4^2$ for all three elements in B (resp. C) as one of our 4-lines must then contain all three elements of B (resp. C) and, by Lemma 4.1 and the part of Lemma 4.3 already proved for $r=2$, we would then have a common triple of frequency 4 elements in four blocks, which is forbidden. So we are forced to use $7^1 6^1 2^1 6^1$ once and $7^1 6^1 4^2$ twice for B and for C, but now the element in B with the pattern $7^1 6^1 2^1 6^1$ cannot meet either of the elements in C with the pattern $7^1 6^1 4^2$. Hence pattern 4I is not viable and and all the patterns in Table 6 have been eliminated.

For the next set of constructions we will concentrate on where possible patterns can be used. Our basic frequency 4 patterns are $7^2 3^1 4^1$ for A, $7^1 6^1 3^1 5^1$ or $7^1 6^1 4^2$ for $B \cup C$, and $6^2 3^1 6^1$ or $6^2 4^1 5^1$ for D. These are the

Table 7: Patterns of X_2 elements over the last 11 blocks

Case	x	Pattern	No. X_2 pairs	Excess X_2 vs. X_2^c	min. forced
2C	2	$7^2 6^2; 3^2 4^3 5^2$	116	5	2
2E	2	$7^2 6^2; 3^3 4^1 5^3$	117	3	1
3B	3	$7^2 6^2; 3^1 4^4 5^2$	119	7	2
3C	3	$7^2 6^2; 3^2 4^2 5^3$	120	5	1
3J	3	$7^2 6^2; 3^3 4^1 5^2 6^1$	122	1	0
4A	4	$7^2 6^2; 4^5 5^2$	122	9	2
4B	4	$7^2 6^2; 3^1 4^3 5^3$	123	7	1
4D	4	$7^2 6^2; 3^2 4^1 5^4$	124	5	1
4G	4	$7^2 6^2; 3^2 4^2 5^2 6^1$	125	3	0
5A	5	$7^2 6^2; 4^4 5^3$	126	9	1
5B	5	$7^2 6^2; 3^1 4^2 5^4$	127	7	1
5G	5	$7^2 6^2; 3^2 4^1 5^3 6^1$	129	3	0

frequency 4 patterns that contribute no extra X_2 vs. X_2^c pairs. Changing an n-line to an $(n-1)$-line in one of the basic patterns gives one extra X_2 vs. X_2^c pair. Our next concern is the two basic patterns for $B \cup C$, say α, β in some order. If both patterns appear in some subset of $B \cup C$ of size 4 or more and $|A| = 4$ (note that $|A| = 4$ for all remaining patterns except possibly 5A), then not all points of $B \cup C$ can meet each other. With a 4 element subset, we have members of the subset in both B and C, so if pattern α appears in B, then we cannot meet a pattern β element in C, so only α is allowed in the subset in C and now, in a similar fashion, only α is allowed in the subset in B.

For pattern 2E: since $|A| = 4$, we need both frequency 5 elements in A. We can allow 3 extra X_2 vs. X_2^c pairs, and the two frequency 5 elements each generate at least one extra pair (with the pattern $7^2 5^3$). We can use at most 6 incidences on 5-lines with elements of A, at most 6 incidences with elements of $B \cup C$, and at most 3 incidences with elements of D. Since we have 15 such incidences, we must use the pattern $7^2 5^3$ for the two frequency 5 elements of A and the pattern $7^1 6^1 3^1 5^1$ for the six elements of $B \cup C$; the remaining basic patterns are $7^2 3^1 4^1$ twice and $6^2 4^1 5^1$ three times, except that a non-basic pattern with a 3-line in place of a 4-line is used once somewhere in these last 5 elements. So let us consider the patterns in B and C, bearing in mind Lemma 4.1 and the part of Lemma 4.3 already proved for $r = 2$ and the fact that a common triple of frequency 4 elements in four blocks is forbidden. Suppose we label our 3-lines as a, b, and c and our 5-lines as d, e and f. We place our first C element in ad, say. Now we cannot place all 3 elements of B in a, nor in d as we would generate a common triple, so we use ae, ae, and bd. Finally we place the other two elements of C in be and be. The incidences are $a^3 b^3 c^0; d^2 e^4 f^0$, and the arrangement is essentially unique. Now the 5-line e already has two incidences of frequency 5 elements of A, so there is no room for the 4 incidences from elements of $B \cup C$, and pattern 2E is not viable.

Consider pattern 2C: since $|A| = 4$, we need both frequency 5 elements in A. The basic patterns $7^2 4^1 5^2$ twice, $7^2 3^1 4^1$ twice, $6^2 4^1 5^1$ three times for the elements of A and D use 4 of the 5 available extra X_2 vs. X_2^c pairs, and leave the incidences of $7^6 6^6 3^4 4^5 5^3$ for $B \cup C$, although somewhere an X_2 vs. X_2^c pair still needs to be placed. So for $B \cup C$ we have either 3 or 4 incidences on 5-lines, and so 3 or 4 instances of the pattern $7^1 6^1 3^1 5^1$ and, after allowing for the various possibilities for the 10 or 11 (nearly) established patterns, we always have an average of more than 1 4-line in the remaining points, so we must have the pattern $7^1 6^1 4^2$ at least once. Now we have a subset of $B \cup C$ of size at least 4 containing both of the patterns $7^1 6^1 3^1 5^1$ and $7^1 6^1 4^2$, and we have already shown this is an impossible situation.

For pattern 3B: we have $|A| = 4$ and our frequency 5 elements must use up at least 6 of the 7 allowed excess X_2 vs. X_2^c pairs. So our frequency 5 elements come from $A \cup B \cup C$ with patterns $7^2 4^1 5^2$, $7^2 4^2 5^1$ and $7^1 6^1 4^1 5^2$ and excesses of 2, 3 and 3 respectively. We must have at least two frequency 5 elements in A. The non-excess frequency 4 patterns are $7^2 3^1 4^1$ for A, $7^1 6^1 4^2$ and $7^1 6^1 3^1 5^1$ for $B \cup C$, and $6^2 4^1 5^1$ for D. There are at least five frequency 4 elements in $B \cup C$ using non-excess patterns, since we can only have at most one element with excess pairs in $B \cup C$ (either a frequency 4 element, or a frequency 5 element with pattern $7^1 6^1 4^1 5^2$). The elements of D and the frequency 5 elements use either eight or nine of the incidences on the 5-lines, so there are one or two uses of the pattern $7^1 6^1 3^1 5^1$ in $B \cup C$, and so at least three uses of the pattern $7^1 6^1 4^2$ in $B \cup C$, and we have shown we cannot mix four or more frequency 4 elements using both patterns.

For pattern 3J: we have $|A| = 4$, so there must be three frequency 5 elements in $A \cup B \cup C$ with patterns $7^2 5^2 6^1$, $7^2 4^1 5^1 6^1$ or $7^1 6^1 5^2 6^1$. This uses up 3 incidences on the 6-line which is in the last 7 blocks. The only other element using a 6 incidence is from D using $6^2 3^1 6^1$. There must be exactly 3 of them using up 3 incidences on the 3-lines. For the frequency 4 elements in A, the patterns are $7^2 3^1 4^1$ and $7^2 3^2$ and our only options for the frequency 4 elements in $B \cup C$ are the patterns $7^1 6^1 3^1 5^1$ and $7^1 6^1 3^1 4^1$. So, the seven frequency 4 elements in $A \cup B \cup C$ (which each need at least one incidence on a 3-line) and the three elements of D (which each need exactly 1 incidence on a 3-line) use up at least 10 incidences on the 3-lines, but only 9 are available.

For pattern 3C: we have $|A| = 4$, so must have at least two frequency 5 elements in A. If we avoid using the pattern $7^1 6^1 4^2$, we can argue as in case 3J that we need at least seven incidences on a 3-line, and only six are available, thus we must use the pattern $7^1 6^1 4^2$ for at least one element in $B \cup C$. Now our frequency 5 elements can each use at most three incidences on a 5-line, and our frequency 4 elements in D can each use at most one incidence on a 5-line, but this still leaves at least three incidences on 5-lines, so we must use the pattern $7^1 6^1 3^1 5^1$ for at least three elements in $B \cup C$, so we now have the situation where we have a subset of size four or more in $B \cup C$ containing both of the patterns $7^1 6^1 3^1 5^1$ and $7^1 6^1 4^2$ which we earlier showed was an impossible situation.

For pattern 4D: we have $|A| = 4$, and since we can only allow 5 extra X_2 vs. X_2^c pairs, we must use the pattern $7^2 5^3$ for at least three of our four frequency 5 elements, using up three of our excess X_2 vs. X_2^c pairs; the fourth frequency 5 element must also use up at least one of the two remaining extra X_2 vs. X_2^c pairs, so we have the pattern $6^2 4^1 5^1$ at least twice, and there is only one 4-line, so Lemma 4.1 forces the same 5-line for these two D elements. We cannot put a third $6^2 4^1 5^1$ pattern on the 4-line, or we would get a common triple in four blocks, so we must have the pattern $6^2 3^1 5^1$ (with an extra X_2 vs. X_2^c pair), and pattern $7^2 5^3$ for our last A element, so we've used all the allowed extra X_2 vs. X_2^c pairs, and there's no way to fill the last two incidences on the 4-line.

For pattern 5B: since we can only allow 7 extra X_2 vs. X_2^c pairs, we must use the pattern $7^2 5^3$ for at least three of our five frequency 5 elements, using up three of our excess X_2 vs. X_2^c pairs. Note we must have $|A| = 4$, so at least one of our frequency 5 elements must be in $B \cup C \cup D$ and have at least two extra X_2 vs. X_2^c pairs. We must have at least two elements with pattern $6^2 4^1 5^1$. Since the pattern $7^2 3^1 4^1$ would force us to use three patterns of $6^2 4^1 5^1$, all on a common 4-line, so the last element in A has frequency 5. Now the basic patterns $7^2 5^3$ four times, $7^1 6^1 5^3$ or $6^2 5^3$ once and $6^2 4^1 5^1$ three times leave $3^3 4^5 5^2$ incidences in the last seven lines for 5 $B \cup C$ points, with at most one extra X_2 vs. X_2^c pair allowing some variability in the basic patterns. So for at least two $B \cup C$ points we must use the pattern $7^1 6^1 3^1 5^1$, and for at least two $B \cup C$ points we must use the pattern $7^1 6^1 4^2$. But these two basic patterns do not meet in the last 7 blocks and cannot all meet in blocks $B_1 - B_4$.

For pattern 5A: we note that we cannot have any frequency 4 elements in A, as there's no pattern that meets the 15 elements of X_2^c, so we must have $|A| = 4$, as $|A| = 3$ needs two frequency 5 elements in each of B and C, and $|A| = 5$ needs two frequency 5 elements in D. Note we can only allow 9 extra X_2 vs. X_2^c pairs, and the pattern $7^2 5^3$ uses up one of our excess X_2 vs. X_2^c pairs, so we must have at least one pattern of $7^2 5^3$. Consequently, we must use the pattern $6^2 4^1 5^1$ for the 3 elements of D. Since we have exactly 15 incidences on 5-lines, the frequency 5 elements must use 12 of these, and we have a total of 4 substitutions (4 for 5 three times and 6 for 7 once) in the basic pattern of $7^2 5^3$ for the frequency 5 elements, using up all the excess X_2 vs. X_2^c pairs. We must have five patterns of $7^1 6^1 4^2$, and for all the B elements to meet the C elements, the frequency 5 element in say B must have the pattern $7^1 6^1 4^2 5^1$ or $7^1 6^1 4^3$. Now let us consider the elements of X_2^c. The third element in $B_i \cap B_j$ lies on a 7-line and a 6-line. To meet the pattern $7^2 5^3$, the other elements of X_2^c on 6-lines must also lie on a 5-line. Now consider the elements of X_2^c that meets the elements of B on a 6-line. How can they meet the 3 elements of C? At most two of these elements could meet C on a 7-line, but there is at least one element of X_2^c

that must meet them on a 4-line, so the three frequency 4 elements in C lie on 3 common lines (a 7- 6- and 4-line) so, by Lemma 4.1 and the part of Lemma 4.3 already proved, they lie on another common 4-line, and so generate too many repeated pairs amongst themselves. Thus, pattern 5A is not viable.

For pattern 4A: we again have $|A| = 4$ and since there is no viable pattern for a frequency 4 element in A, the frequency 5 elements are all in A. So the basic pattern for elements in A is $7^2 4^1 5^2$, for $B \cup C$ is $7^1 6^1 4^2$, and for D is $6^2 4^1 5^1$. This accounts for 8 of the 9 allowed extra X_2 vs. X_2^c pairs, and the last pair is given by changing a 5-line into a 4-line to give a single non-basic pattern for some element in $A \cup D$. If this non-basic pattern were in D, then the A elements would use 4 incidences on each 5-line, so the two pattern $6^2 4^1 5^1$ elements in D could not coincide, and obviously neither can coincide with the $6^2 4^2$ pattern, but we have shown that 3 frequency four elements on the same two lines must have a coincident pair. Thus we must conclude A has one element with the non-basic pattern $7^2 4^2 5^1$. This element will meet the coincident D pair on a 4-line, and meet the third element of D on a 5-line. Now lets consider the possible X_2^c patterns. If $B_i \cap B_j$ contains three frequency 4 elements, then the basic pattern is 7-6 three times, 7-4-4 once, 6-4-4 three times, 4-4-4-4 once and 4-4-5 seven times, except that for one element we must substitute a 5-line for a 4-line. If $B_i \cap B_j$ contains a frequency 5 element, then the basic first pattern is 7-6 twice, 7-4-4 twice, 6-4-4 four times, and 4-4-5 seven times, except that for one element we must substitute a 5-line for a 4-line; the second basic pattern is 7-6 twice, 7-6-4 once, 7-4-4 once, 6-4-4 three times, and 4-4-5 eight times. As with pattern 2E, the distribution of the $B \cup C$ incidences over the 4-lines is $3^2 4^1 2^1 0^1$, and to meet all elements of $B \cup C$, a frequency 4 element of X_2^c must lie on a 6-line and a 7-line, or else two 4-lines (more specifically, the two 4-lines must be those with either 3^2 or $4^1 2^1$ $B \cup C$ incidences). Now consider an X_2^c element with pattern 6-4-4. Suppose this meets the elements of C on the 6-line. Now it can meet the elements of B on two 4-lines, but on each of those 4-lines where it meets an element of B, it also meets at least one element of C, so this pattern can meet at most 12 distinct elements of X_2, and there is no viable set of patterns for X_2^c that avoids this 6-4-4 pattern, and so no viable pattern 4A.

For pattern 4B: since we can only allow 7 extra X_2 vs. X_2^c pairs, we must use the pattern $7^2 5^3$ for at least one of our four frequency 5 elements, using up one of our excess X_2 vs. X_2^c pairs. Note we must have $|A| = 4$. Next, we want to show we must have the pattern $7^1 6^1 4^2$ for at least 3 elements of X_2. Note that in $B \cup C$ all frequency 4 patterns, whether basic or not, use a 3-line if the pattern $7^1 6^1 4^2$ is forbidden. Suppose we replace a frequency 4 element in $B \cup C$ with a frequency 5 element from A. This will add a $7^2 3^1 4^1$ element to A, so each $7^1 6^1 4^2$ pattern avoided costs an incidence on the 3-line, so we can avoid the pattern at most three times. Next we look at the X_2^c elements. Each must use either a 7-line or a 5-line, and the patterns 7-3-4 and 6-3-5 each have an extra X_2 vs. X_2^c pair which is the minimum for

X_2^c elements with just one incidence in blocks B_1–B_4. For the frequency 5 element, the pattern 7-3-4-4, 6-3-4-5, and 3-4-4-5 generate 5, 5 and 3 extra pairs, so we see the only way we can achieve 7 or fewer extra pairs and fill the 6- and 7-lines is to have every element in $B_i \cap B_j$ appear on both a 6-line and a 7-line. We can easily deduce that the X_2^c patterns must be 7-6 twice (for elements 1 and 2), 6-3-5 three times, 7-3-4 once, 4-4-5 six times, 3-5-5 once, plus either 7-6 once and 3-4-4-5 once, or 7-6-3 once and 4-4-5 once. In either case, we have an instance of the X_2^c pattern 3-5-5, and this element cannot meet the X_2 elements with pattern $7^1 6^1 4^2$, so pattern 4B is not viable.

For pattern 4G: since we can only allow 3 extra X_2 vs. X_2^c pairs, we must use the pattern $7^2 5^2 6^1$ for at least one frequency 5 element of X_2. Now let us consider the elements of X_2^c. To meet the pattern $7^2 5^2 6^1$, every element of X_2^c must lie on a 7-line or else a 5- or 6-line in the last seven blocks. If the third element in $B_i \cap B_j$ has frequency 5, then we have the patterns 7-3-3 twice, and 6-3-5 four times. Each of the last 4 elements has an extra X_2 vs. X_2^c pair, and this is the minimum possible way of using up the incidences on blocks B_1–B_4, but still causes too many X_2 vs. X_2^c pairs. The alternative of replacing a 7-3-3 or a 6-3-5 or both by 7-3-3-4 or 6-3-3-5 or 7-6-3-3 just makes matters worse, so we must have the third element in $B_i \cap B_j$ has frequency 4 and lies on a 7-line and a 6-line. Then the frequency 5 element of X_2^c must have pattern 3-3-4-5, and we must also have 7-3-3 once and 6-3-5 three times (causing 5 extra X_2 vs. X_2^c pairs). These are the patterns with the smallest number of extra pairs and are still too many.

For pattern 5G: since we can only allow 3 extra X_2 vs. X_2^c pairs, we must use the pattern $7^2 5^2 6^1$ for at least one frequency 5 element of X_2. Now let us consider the elements of X_2^c. To meet the pattern $7^2 5^2 6^1$, every element of X_2^c must lie on a 7-line or else a 5- or 6-line in the last seven blocks. Now the third element in $B_i \cap B_j$ has frequency 4 and lies on a 7-line and a 6-line and we must also have the patterns 7-3-3 once and 6-3-5 three times; these are the patterns with the smallest number of extra pairs and together use all three of the extra X_2 vs. X_2^c pairs. Now using the last 6-line, we must have 3-4-6 three times, and so 3-5-5 four times and 4-4-5 once. But there is only one 4-line, so the final pattern needed does not exist.

■

Now the proof of Lemma 4.3 is complete.

6 C(28, 9, 2)

In this section, we will finally prove that $C(28, 9, 2) \geq 14$. Recall that we have already proved in Lemma 4.4 and Lemma 4.6 that in a $(28, 9, 2)$-Covering Design on 13 blocks, there are no 0-blocks and that any two blocks intersect in at most 5 elements. Also we showed in Lemmas 4.1 and 4.3 that if a pair of frequency 4 elements occur together in B_i and B_j, then that pair occurs together in 4 blocks provided $|B_i \cap B_j| \geq 3$; our first step is to show that the condition $|B_i \cap B_j| \geq 3$ can be dispensed with.

Lemma 6.1 *In a* $(28, 9, 2)$*-Covering Design with* 13 *blocks, any pair of frequency* 4 *elements occurs together in either one or four blocks.*

Proof: Assume that elements 1 and 2 have frequency 4.
Case 1: First we deal with the possibility that elements 1 and 2 occur together in exactly three blocks. Then this is the situation:
$B_i : \{1, 2, \ldots\}$
$B_j : \{1, 2, \ldots\}$
$B_1 : \{1, 2, \ldots\}$
$B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$

Now $|B_i \cup B_j \cup B_1| \leq 2 + 3(9 - 2) = 23$, so $|B_2 \cap B_3| \geq 28 - 23 = 5$ in order that elements 1 and 2 meet all the other elements. By Lemma 4.6, we must have $|B_2 \cap B_3| = 5$, and so the elements (other than 1 and 2) in B_i, B_j, B_1, and $B_2 \cap B_3$ are all distinct. Now $B_2 \cap B_3$ contains at most two elements of frequency 4 (or else by Lemmas 4.1 and 4.3 we will have a triple (or larger) of frequency 4 elements occurring together in four blocks, generating too many extra pairs). Since B_i, B_j and B_1 are at least 1-blocks, we need at least 6 elements of frequency more than 4, but at most 5 exist.
Case 2: Now we have to eliminate the possibility that a pair occurs together in exactly two blocks. Assume that elements 1 and 2 have frequency 4 and occur in the intersection of B_i and B_j. Further assume that they do not occur together again in any other block. Then this is the situation:
$B_i : \{1, 2, 3, 4, \ldots, 9\}$
$B_j : \{1, 2, 10, 11, \ldots, 16\}$
$B_1 : \{1, \ldots\}$
$B_2 : \{1, \ldots\}$
$B_3 : \{2, \ldots\}$
$B_4 : \{2, \ldots\}$

Let X_2 denote the 12 elements 17–28. Now $X_2 \subset B_1 \cup B_2$ and similarly $X_2 \subset B_3 \cup B_4$. Now if any of $B_1 \cap B_3$, $B_1 \cap B_4$, $B_2 \cap B_3$ or $B_2 \cap B_4$ contain more than two frequency 4 elements, then by Lemmas 4.1 and 4.3 we will generate too many extra pairs, so these four intersections contain at most 8 frequency 4 elements. Also the union of these four intersections contains X_2, so X_2 contains at most 8 frequency 4 elements, and so at least 4 frequency 5 elements. This implies $B_i \cup B_j$ contains at most one frequency 5 element, so at least one of B_i and B_j must be a 0-block, violating Lemma 4.4. Thus a pair of frequency 4 elements cannot appear in exactly two blocks.

Since we have a covering design, every pair occurs together at least once, and now the only viable possibilities are those given in the statement of the lemma. ■

Lemma 6.2 *Let $(\mathcal{X}, \mathcal{A})$ be a PBD with $|\mathcal{X}|$ points, all of frequency 4, and with at most 13 blocks. Then $|\mathcal{X}| \leq 13$. Furthermore, if $|\mathcal{X}| = 13$, then the PBD is $PG(2,3)$, and if $|\mathcal{X}| = 12$, then the PBD is a punctured $PG(2,3)$ (i.e., $PG(2,3)$ with a point removed).*

Proof: If $|\mathcal{X}| = 14$, then 56 incidences on 13 blocks generates at least 94 pairs which exceeds $\binom{14}{2} = 91$, and 56 incidences on fewer blocks generates even more pairs, thus we cannot have $|\mathcal{X}| = 14$. If $|\mathcal{X}| > 14$, we can delete all but the first 14 points, and this process normally converts a PBD on $|\mathcal{A}|$ blocks into a PBD, possibly on fewer blocks with some points having a smaller frequency. However, if we retain all "blocks" of size 1 in the conversion process, we do still get a PBD on at most $|\mathcal{A}|$ blocks with all points having frequency 4, and we know this latter PBD does not exist, so no PBD with $|\mathcal{X}| > 14$ can exist either. If $12 \leq |\mathcal{X}| \leq 13$, then the unique distribution of block sizes which generates the minimum number of pairs on $4|\mathcal{X}|$ incidences generates exactly $\binom{|\mathcal{X}|}{2}$ pairs. If $|\mathcal{X}| = 13$, then the PBD is a projective plane of order 3, hence PG$(2,3)$, and if $|\mathcal{X}| = 12$, then adding a point to the short blocks produces PG$(2,3)$. ■

Now what Lemma 6.1 shows is that the SFs defined by the elements of frequency 4 really behave like points in a PBD, and by considering the design generated by the SFs, we have converted the consideration of a covering design into the consideration of a pairwise balanced design. Now we have at least 23 points of frequency 4, and the largest size possible for an SF is 2 (for a $(28,9,2)$-Covering Design with 13 blocks), so we have at least 12 SFs, and now Lemma 6.2 describes the possible PBDs for the design of the SFs. We are now ready to prove our main result of this section.

Theorem 6.3 $C(28,9,2) \geq 14$.

Proof: Suppose the contrary and we have a $(28,9,2)$-Covering Design with 13 blocks. Let s be the number of elements of frequency 5 or more. These elements have a total of $13(9) - 4(28 - s) = 4s + 5$ incidences and, since there are no 0-blocks, we have $4s + 5 \geq 13$, i.e., $s \geq 2$.

Now assume there are 13 distinct SFs. By Lemma 4.10, every element of frequency 5 must coincide with some element of frequency 4 for 4 of its incidences. However if a frequency 4 element meets a common pair of other elements on all four of its blocks, then it has 6 extra pairs and only 5 are needed. Hence, we can only have a frequency 5 element that always occurs with some $SF(\alpha)$ if $|SF(\alpha)| = 1$. So every frequency 5 element is paired with a unique $SF(\alpha)$ of size one. So the maximum number of frequency 4 and frequency 5 elements is at most 26. This leaves at least two elements that have frequency greater than 6. The only two cases that accommodate this restriction are $s = 2$ where there is 1 element of frequency 6 and 1 element of frequency 7 or else $s = 3$ with 2 elements of frequency 6 and 1 element of frequency 5. In both cases, after the frequency 4 and 5 elements are placed in the partial design, there is at most one unused incidence per block, and there is no way the two elements of frequency larger than 5 can meet. So the case of 13 distinct SFs is impossible.

Hence we must have 12 distinct SFs. It will be convenient to think of their underlying PBD as $PG(2,3)$ with its puncture point being replaced by a "phantom" point. We have at most 24 frequency 4 points in SFs, so $s \geq 4$.

Suppose $s = 5$, so we have 5 elements of frequency 5. We can pair one frequency 5 element with the unique frequency 4 element whose SF has size 1, and we can have 3 frequency 5 elements coincide with the phantom point in all blocks it occurs in. We cannot have more than 3 such frequency 5 elements in the covering design, as 3 of these blocks have 6 (i.e., 3 pairs of) frequency 4 points, and the other has 5 (its sixth place will be occupied by the earlier frequency 5 element we placed). Since the partial design now has all its blocks which contain the trio of frequency 5 elements of size 9, the last frequency 5 element can not occur with any of trio of frequency 5 elements.

Now suppose $s = 4$, so we have 3 elements of frequency 5, 1 element of frequency 6 and all SF's have size 2. Since our frequency 5 elements must occur in every occurrence of some frequency 4 element but not a pair of frequency 4 elements, our only option for those frequency 5 elements is to have the 3 frequency 5 elements occur in every block containing the phantom point. At this stage we have still to place three last elements of our frequency 5 elements and the frequency 6 element. However, in the covering design we have now formed 4 complete blocks, and the other 9 have only one free incidence per block, so there is no way our frequency 6 element can meet any frequency 5 elements.

So no $(28, 9, 2)$-Covering Design with 13 blocks exists, and our result follows. ■

7 Constructions

In this section, we establish the upper bounds for the following three infinite families of covering designs:
1) $(13r + 2, 4r + 1, 2)$-Covering Designs,
2) $(13r + 3, 4r + 1, 2)$-Covering Designs,
3) $(13r + 6, 4r + 2, 2)$-Covering Designs.

We do this using a construction similar to the construction described at the end of Section 1. First we need an input design. It is almost a covering design.

Lemma 7.1 *There exists* 19 *elements in* 14 *blocks so that every pair of elements occurs at least once.*

Proof:

1 2 3 4 16 19	3 5 10 12 14 15
1 5 6 7 14 15	3 6 8 13 16 17
1 8 9 10 17 19	3 7 9 11 16 18
1 11 12 13 18 19	4 5 9 13 14 15
2 5 8 11 14 15	4 6 10 11 16 17
2 6 9 12 16 17	4 7 8 12 17 18
2 7 10 13 18 19	5 6 14 15 16 17 18 19

■

Note that all blocks have size 6 except the last one which has size 8. Also note that the elements 1 to 13 form the projective plane on the first 13 blocks. The pair $\{5, 6\}$ occurs one extra time in the larger fourteenth block.

Theorem 7.2 $C(13r + 6, 4r + 2, 2) = 14$ *for* $r \geq 2$.

Proof: Let $r \geq 2$. Consider the blocks in Lemma 7.1. Replace each element $i, 1 \leq i \leq 13$ by r copies of itself, $i_1, i_2, \ldots, i_r$ wherever it occurs in the covering design. Add any distinct $2r - 4$ elements to the last block. The resulting design is a $(13r+6, 4r+2, 2)$-Covering Design. So $C(13r+6, 4r+2, 2) \leq 14$ for $r \geq 2$. Using Theorem 3.5, we have that $C(13r+6, 4r+2, 2) \geq 14$. ■

To construct the other two infinite families, we need a slightly different starting configuration.

Lemma 7.3 *There exists* 16 *elements in* 14 *blocks so that every pair of elements occurs at least once.*

Proof:

1 2 3 4 15	3 5 10 12 14
1 5 6 7 14	3 6 8 13 15
1 8 9 10 16	3 7 9 11 15
1 11 12 13 16	4 5 9 13 14
2 5 8 11 14	4 6 10 11 15
2 6 9 12 15	4 7 8 12 16
2 7 10 13 16	3 5 6 14 15 16

■

Note that all blocks have size 5 except the last one which has size 6. Also note that the elements 1 to 13 form the projective plane on the first 13 blocks. The elements 3, 5 and 6 occur one extra time in the larger fourteenth block.

Theorem 7.4 *$C(13r + 3, 4r + 1, 2) = 14$ for $r \geq 2$.*

Proof: Let $r \geq 2$. Consider the blocks in Lemma 7.3. Replace each element $i, 1 \leq i \leq 13$ by r copies of itself, $i_1, i_2, \ldots, i_r$ wherever it occurs in the covering design. Add any distinct $r - 2$ elements to the last block. The resulting design is a $(13r+3, 4r+1, 2)$-Covering Design. So $C(13r+3, 4r+1, 2) \leq 14$ for $r \geq 2$. Using Theorem 2.1, we have that $C(13r+3, 4r+1, 2) \geq 14$. ■

Theorem 7.5 *$C(13r + 2, 4r + 1, 2) = 14$ for $r \geq 3$.*

Proof: Consider the covering design constructed in Theorem 7.4. One element can be deleted from the design and all the short blocks can have an element added to them to construct a $(13r+1, 4r+1, 2)$-Covering Design. So $C(13r + 2, 4r + 1, 2) \leq 14$ for $r \geq 2$. Using Theorem 2.1, we have that $C(13r + 3, 4r + 1, 2) \geq 14$ for $r \geq 3$. ■

8 Conclusion

Putting it all together we get the following theorem.

Theorem 8.1 *$C(v, k, 2) = 13$ for $3k < v \leq 13k/4$ except for the following three infinite cases:*
1) $C(13r + 2, 4r + 1, 2) = 14$ for $r \geq 2$;
2) $C(13r + 3, 4r + 1, 2) = 14$ for $r \geq 2$;
3) $C(13r + 6, 4r + 2, 2) = 14$ for $r \geq 2$;
and except for the following small cases:
$C(19, 6, 2) = 15$ and $C(16, 5, 2) = 15$.

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