Friendship 3-Hypergraphs

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1 Introduction

A *friendship* graph is a graph in which any two vertices have exactly one common neighbour.

There is a beautiful proof that characterizes all such graphs as windmill graphs (i.e. triangles all joined at a common vertex or hub). The center vertex is often referred to as the politician who is friends with everyone. The graph is called the *universal friend graph*. It exists only for an odd number of vertices.

Since that solves the problem, the problem has been generalized many ways.

We are interested in the generalization that Sós made:

A *Friendship 3-Hypergraph* is a 3-hypergraph in which any 3 vertices (elements), u, v and w, occur in pairs with a unique fourth element x; i.e., uvx, uwx, vwx are 3-hyperedges. The element x is said to *complete* the elements u, v and w.

How many of the results from graph theory are going to generalize to 3-hypergraphs?

Is it going to be more like graph theory or more like design theory?

First Off - Is there a universal friend 3-hypergraph? And if so what does it look like?

It looks like a Steiner Triple System in which each pair of the Steiner Triple System also occurs in a triple with a new spe4cial element. Call this the *universal friend 3-hypergraph.*

Eg. For n=8.

013 124 235 346 450 561 602

∞01 ∞02 ∞03 ∞04 ∞05 ∞06 ∞12

∞13 ∞14 ∞15 ∞16 ∞23 ∞24 ∞25

∞26 ∞34 ∞35 ∞36 ∞45 ∞46 ∞56

Theorem(Sós) For $n = 2, 4 \bmod 6$, there exists a universal friend 3-hypergraph.

Proof: For $n = 1, 3 \bmod 6$, there exists a Steiner Triple System (STS). To the STS add the triples ∞ij where $0 \leq i \neq j \leq n-2$. Clearly, ∞ completes i, j, k as we have ∞ij, ∞ik and ∞jk. Also to find what completes ∞, i and j, find the triple containing ij. Say it is ijk. Then we have ∞ik, ∞jk and ijk. Because every pair ij occurs exactly once in an STS, there are no other completions. $\square$

Sós asked whether there were any other friendship 3-hypergraphs?

2 History

Hartke and Vandenbussche answered this in the affirmative. They formulated the problem as an integer programming problem. Using CPLEX to solve the integer program, they found Friendship 3-Hypergraphs on 8 vertices (unique), 16 vertices(≥ 3 non-isomorphic hypergraphs) and 32 vertices (≥ 1 non-isomorphic hypergraphs). The hypergraph was regular; i.e., all vertices appeared the same number of times. Further the 3 friendship graphs at 16 vertices had 108, 114 and 272 hyperedges (or triple).

Navin saw this and said these friendship graphs come from a geometry.

Theorem (H&V) Every hyper-edge must be contained in a unique K_4^3.

This means the triples of the friendship graph can be partitioned into K_4^3's or what we call quads or 4-sets. This makes it seem more like design theory. We call this a friendship design with elements in quads.

eg. n=8 b=7

∞013 ∞124 ∞235 ∞346 ∞450 ∞561 ∞602

eg. n=8 b=8

0123 0145 0167 0246 1357 2345 2367 4567

or

0123 4567

0145 2367

0167 2345

0246 1357 planes of AG(2,3)

Think of each element in binary (low order bits on the right).

Let F be a field of order 2 and let V be a vector space of dimension $n+1$; i.e., $n+1$ tuples. There are 2^{n+1} vectors. $P(V)$ is the projective space of dimension n;.i.e., the lines of the vector space are the points of $P(V)$. There are $2^{n+1}-1$ points in $P(V)$. Let H be a hyperplane in $P(V)$. Then H has 2^n-1 points and dimension $n-1$. $A(V)=P(V)\backslash H$ is the affine space of 2^n points and dimension n. A plane in $P(V)$ has 7 points and a plane in $A(V)$ has 4 points. If the points in the affine plane are a,b,c,d then $a+b+c=d$. The other points in the hyperplane of $P(V)$ are $a+b$, $a+c$ and $b+c$. Each set of disjoint planes in $A(V)$ can be associated with the three points that form a line in the hyperplane $P(v)$ that do not occur on the hyperplane in $A(V)$ that goes through point 0. So instead of looking for sets of 2^{n-2} planes in $A(V)$, we will look for lines in the $P(V)$.

Now we want to translate this problem from finding sets of 2^{n-2} planes in the affine plane to finding lines in the projective plane. This can be done.

The subset S of lines obey two properties
1) There is no set of 4 lines in S such that 3 of them form a triangle and the 4'th line consists of the midpoints of the lines in the triangle. Property is symmetrical.

2) For any line, l, not in S there is a unique set of 3 lines in S such that l is a line through their midpoints.
Strange-like property

eg. Suppose we have lines 123, 145 and 246. They form a triangle. Then we can not have line 356 as it is the midpoint of that triangle

1

3 5

2 6 4 this is property 1

Property 2) Now if 356 is not in S then there must be a unique set of 3 lines in S that form a triangle and such that 356 is the midpoint for each side. The 3 lines are not necessarily the ones shown.

0 1 2 3   4 5 6 7   8 9 10 11   12 13 14 15

0 1 4 5   2 3 6 7   8 9 12 13   10 11 14 15

0 1 6 7   2 3 4 5   8 9 14 15   10 11 12 13

:   :   :   :

0 7 11 13   1 6 10 13   2 5 9 14   3 4 8 15

instead of looking for the 68 4-sets above, we look for the 17 3-sets below

1 2 3,   1 4 5,   1 6 7,   1 14 15,   2 4 6,   2 12 14,   2 13 15,
3 8 11,   3 9 10,   4 10 14,   4 11 15,   5 8 13,   5 9 12
6 8 14,   6 9 15,   7 10 13,   7 11 12

So the recursion is 17 deep instead of 68 deep.

So we could search for a set S in the $P(V)$. A regular backtracking program could then easily search the state tree. We found a thousand or so solutions of which 3 were non-isomorphic on 13, 14 and 17 lines. The same ones as found by H & V, but we know there are no others coming from the geometry.

At 5 tuples, the problem is too big too search completely and we only found the one that H & V found.

3 Bounds

Lemma 1 Every pair of elements occurs in at least one quad.

Lemma 2 If n odd, then there are at least $n(n+1)/8$ quads
If n even, there are at least $n^2/8$ quads.
Proof: Count the number of quads that contain an element of a particular pair.

Lemma 3 the number of quads is at most $n(n-1)^2/36$.
Proof: Count the number of quads that contain the pairs of a particular triple.

n	old lower bd $n(n-2)/8$	new lower bd new Lemmata	new upper bd $\frac{\binom{n}{3}(2n-6)}{4(3n-10)}$	old upper bd $\binom{n}{3}/4$	actual # blocks
4	1	2	1	1	1
5	2	3	2	3	-
6	3	5	4	4	-
7	5	6	7	6	-
8	6	8	10	14	7,8
9	8	12	14	21	-
10	10	13*	21	30	12*
11	13	17	28	41	-
12	15	18	38	55	-
13	18	23	49	71	?
14	21	25	62	91	26,?
15	25	34	78	113	?
16	28	32	95	140	35,52,56,68,?
32	120	128	836	1240	155,344,?

Table 1: Bounds and Number of Blocks on Quads in Friendship Designs

*Lemma 3.4 assumes friendship hypergraph is not the universal friend graph.

4 Our Computer Results

Running a straightforward backtracking program, we were able to reproduce all the complete searches that H&V did for $n \leq 10$. For $n = 11$, the straightforward approach is the Halting Problem.

The problem is that the search tree is very bushy. That is there are many choices from each node. Also we don't get to a dead end, until we are many levels down the search tree.

Our Algorithm - 7 Steps

1) For M= Max. # of pairs downto 2 do
the next 6 steps
We assume that that there is a pair that occurs M times and no pair occurs more than M times; i.e.,

n=10 M=4

1234

1256

1278

129t

We call this the starter set,

2)From the starter set, generate all possible sets that contain quads containing a 1 that does not cause two completions for some triple of elements. We will assume that these are the only quads in the set containing element 1. We check to see if every element occurs in a block with element 1.

eg. $n = 8, M = 3$

1234 1256 1278

1234 1256 1278 1357

1234 1256 1278 1358

1234 1256 1278 1357 1368

etc.

3)Eliminate isomorphic copies. Call what is left 1-sets

4)For each of the 1-sets generate a list of “forces”.

eg. $n = 9$ $M = 3$

1234

1256

1278

1357

Pair 13 occurs with elements 2,4,5,7

Pair 16 occurs with elements 2,5

Since there are no more quads with 1 in them, the only completion for 1,3,6 is the element 2. So the pair 36 must occur with a 2 in a quad. So the triple 236 must occur in some quad. The only possibilities are 2346, 2356, 2367, 2368 and 2369. But 2346 has a 3-intersection with the first quad, 2356 has a 3-intersection with the second quad. This leaves 2367 and 2368 and 2369 as forces. That is one of these 3 quads must be in the design. We get a list of these ”forces” for a 1-set. If we can pick a quad from each “force”, then we have a candidate. If not we have a dead end we can eliminate this possibility.

A 1-set may generate 0, 1, 2 or many candidates. All the candidates are grouped together.

We go through the “forces” in order of less dense to most dense. The whole idea is to reduce the bushiness of the search tree.

5)Eliminate isomorphic candidates.

6)Throw the candidates into a normal backtracking program that will see if they lead to solutions. The program starts its search to add on quads at 1234.

7)Eliminate isomorphic solutions.

Time in seconds

n \ M	2	3	4	5
6	0	.	.	.
7	0	.	.	.
8	0	0	.	.
9	0	0	.	.
10	3	14	1	.
11	66	86	11	.
12	1893	16310	17382	816

$n = 8$

Case \ M	2	3	4	5
All 1-sets	24	35	.	.
Non-isomorphic 1-sets	2	6	.	.
All Candidates	0	13	.	.
Non-isomorphic Candidates	0	5	.	.
Solutions	0	2	.	.
Non-isomorphic Solutions	0	2	.	.

$n = 9$

Case \ M	2	3	4	5
All 1-sets	157	482	.	.
Non-isomorphic 1-sets	4	14	.	.
All Candidates	0	6	.	.
Non-isomorphic Candidates	0	3	.	.
Solutions	0	0	.	.
Non-isomorphic Solutions	0	0	.	.

$n = 10$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	9	75	46	.
All Candidates	384	973	359	.
Non-isomorphic Candidates	4	131	108	.
Solutions	0	0	1	.
Non-isomorphic Solutions	0	0	1	.

$n = 11$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	12	461	1045	.
All Candidates	0	170514	18351	.
Non-isomorphic Candidates	0	11830	14504	.
Solutions	0	0	0	.
Non-isomorphic Solutions	0	0	0	.

$n = 12$

Case \ M	2	3	4	5
Non-iso. 1-sets	21	3414	39935	11410
All Candidates	0	71630269	76553127	3105440
Non-iso. Cand.	0	5825458	70819810	2802491
Solutions	0	0	0	0
Non-iso. Sol.	0	0	0	0

5 Conjectures

1)[H&V] There are no friendship 3-hypergraphs on an odd number of points.
2) There are no friendship 3-hypergraphs on $n = 0 \mod 6$ points.
3) No BIBD is a friendship design.
4) There are an infinite number of friendship 3-hypergraphs that are not universal friend 3-hypergraphs.
5) All friendship 3-hypergraphs are either a universal friend 3-hypergraph or are regular.
6) All non-universal friendship graphs are on 2^n points.

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